Lattice Reduction: Problems and Algorithms

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Plan of the talk.

- Mathematical definitions: lattices, lattice invariants, reduction.
- Algorithmic lattice problems: γ-SVP, γ-CVP.
- Lattice algorithms: Gauss, LLL, BKZ.
- Practical lattice reduction.

Usefulness of lattices in computer science.

- Great tool for cryptanalysis.
- Interesting for building cryptosystems.
- Computer algebra: factorization of polynomials over $\mathbb{Z}$.
- Algorithmic number theory: ideals in number fields, small roots of polynomials, minimal polynomials . . .

Basic Geometry of Numbers

A first definition of a lattice.

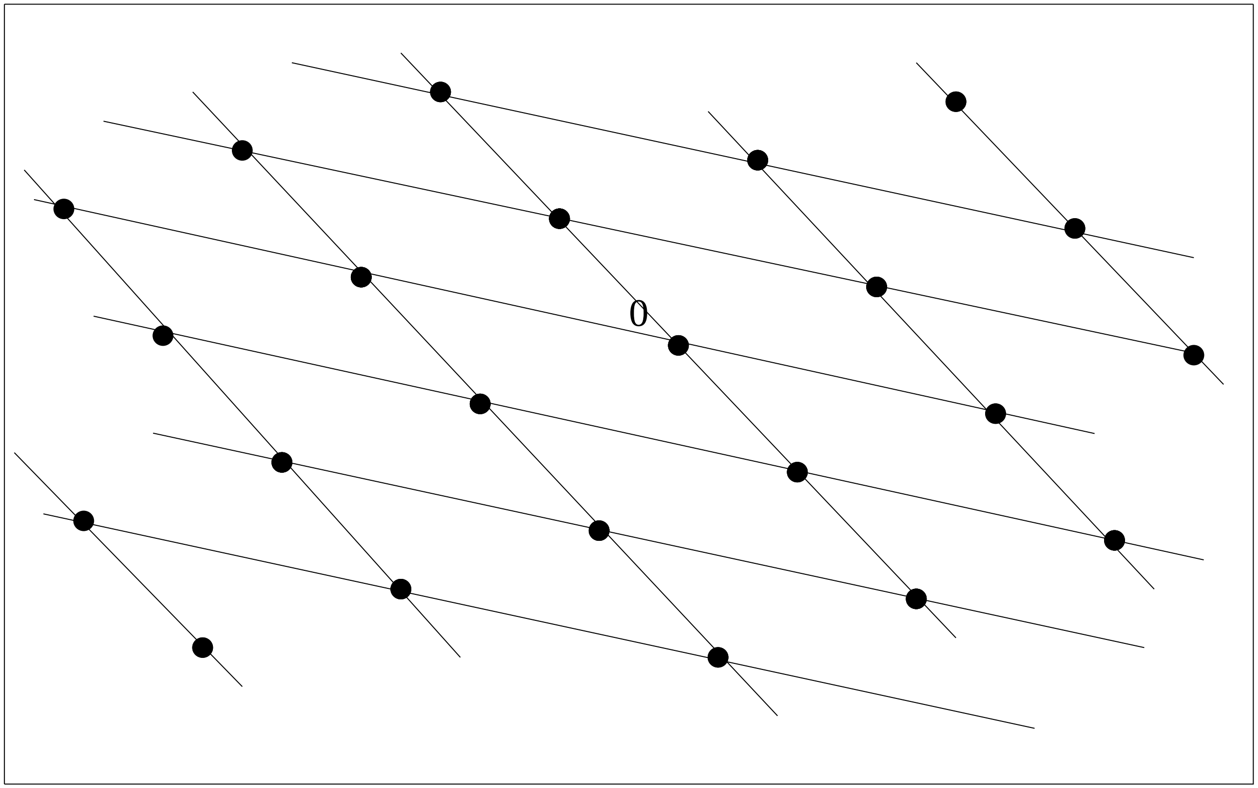
A lattice is a discrete subgroup of a Euclidean space.

- Euclidean space: we are living in $\mathbb{R}^n$.
- Subgroup: 1) $\mathbf{b} \in L \Rightarrow -\mathbf{b} \in L$,
  2) $\mathbf{b}_1, \mathbf{b}_2 \in L \Rightarrow \mathbf{b}_1 + \mathbf{b}_2 \in L$.
- Thus: $\mathbf{0} \in L$, and L is stable by linear integer combinations.
- Discrete: no accumulation point,
  i.e., there is a small open ball containing only $\mathbf{0}$.

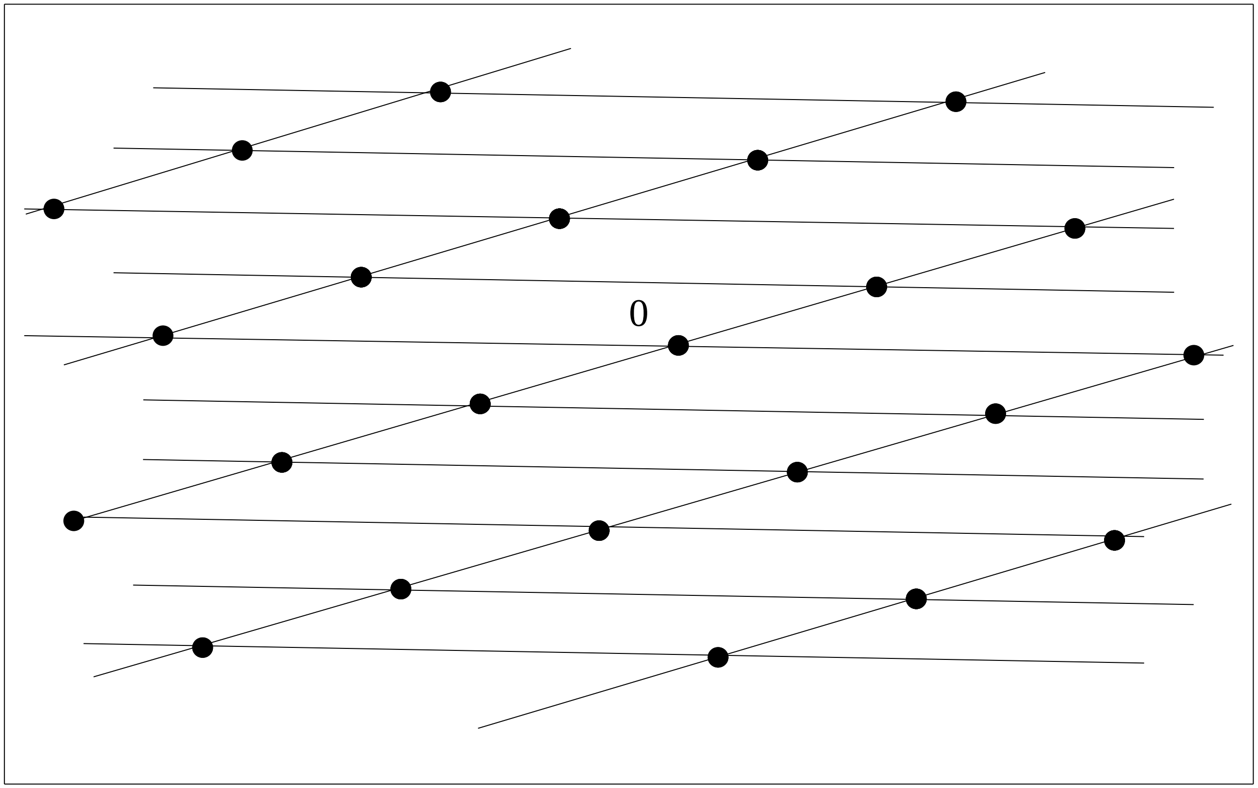
First examples.

- Simplest non-trivial example: $\mathbb{Z} \subset \mathbb{R}$.
- Quite simple too: $\mathbb{Z}^d \subset \mathbb{R}^n$ with $d \leq n$.
- Any subgroup of $\mathbb{Z}^d \subset \mathbb{R}^n$ with $d \leq n$.

A 2-dimensional lattice.



The same lattice.



Second definition of a lattice.

A lattice is the set of all integer linear combinations of some linearly independent vectors in a Euclidean space.

- The two definitions are equivalent.
- $L = \left\{ \sum_{i=1}^{d} x_i \mathbf{b}_i, (x_1, \ldots, x_d) \in \mathbb{Z}^d \right\}$,
  where the $\mathbf{b}_i$'s are linearly independent vectors of $\mathbb{R}^n$.
- Lattice dimension: d.
- Embedding dimension: n.
- $\mathbf{b}_1, \ldots, \mathbf{b}_d$ is a lattice basis. It is not unique.

The second definition is not always the good one.

Let $A = (a_{i,j})_{i,j}$ be an $n \times m$ matrix of integers with $n < m$.

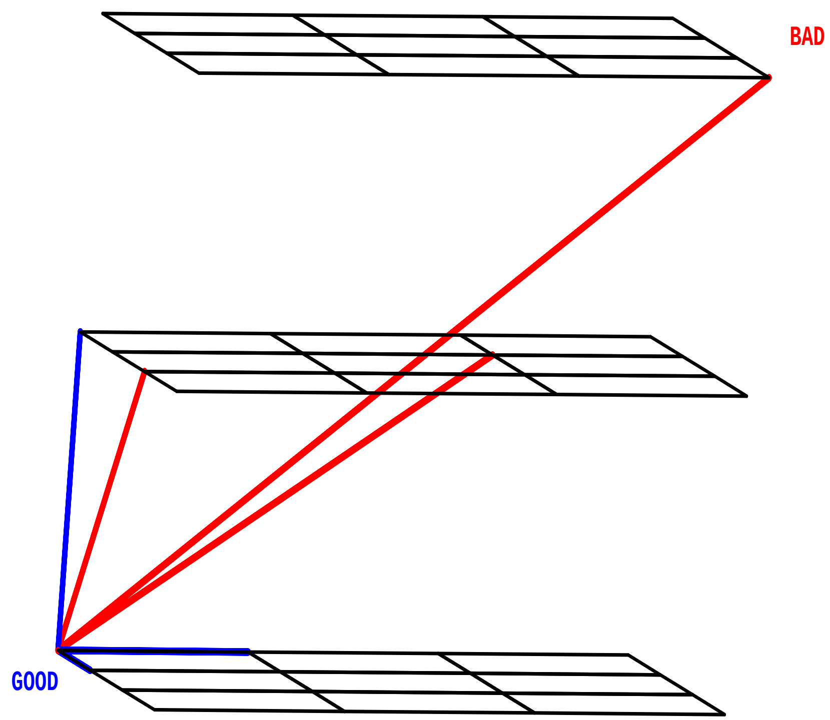
Consider the system of integer equations:

$$\left\{ \begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,m}x_m & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,m}x_m & = & 0 \\ \vdots & & \vdots & & \ldots & & \vdots & & \vdots \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \ldots & + & a_{n,m}x_m & = & 0 \end{array} \right. .$$

The set of solutions $(x_1, \ldots, x_m)$ is a lattice L.

If the rows of A are linearly independent, $\dim(L) = m - n$.

Two bases of a 3-dimensional lattice.



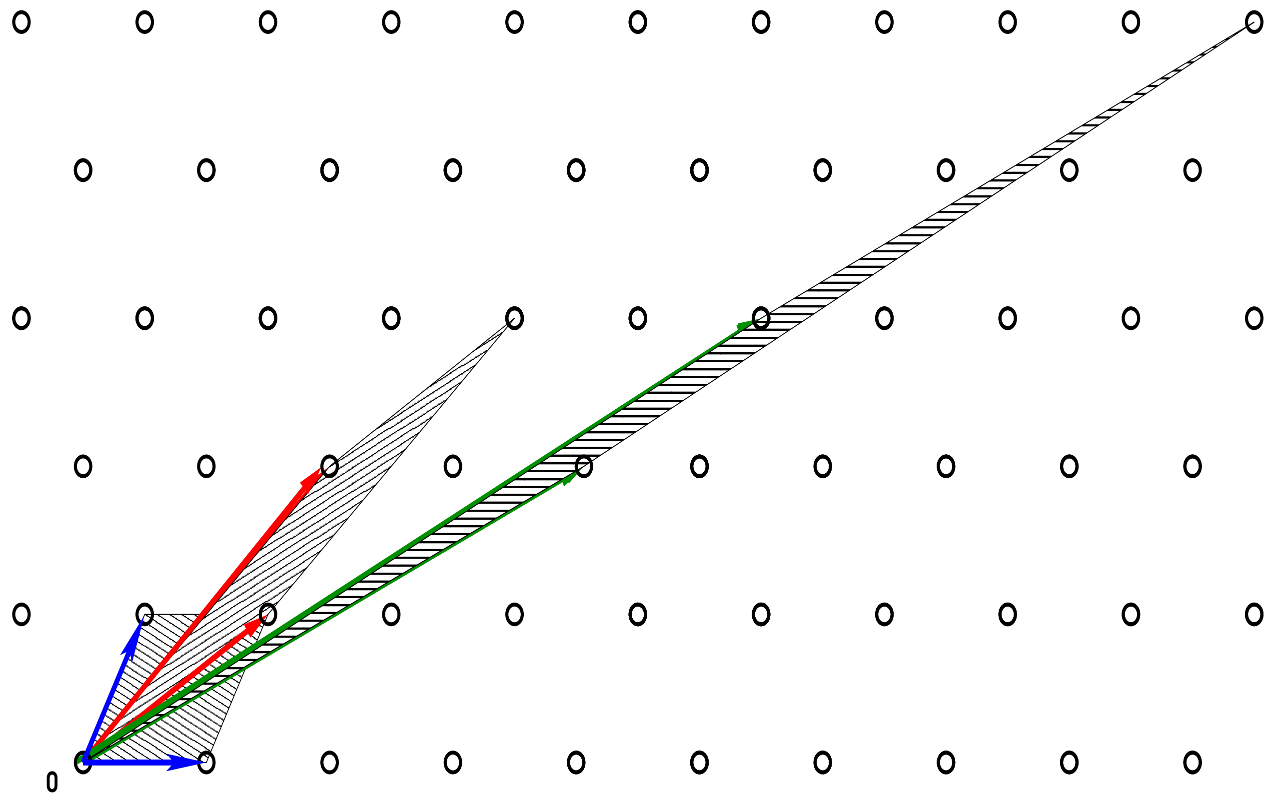
An infinity of bases for a given lattice.

- For a given lattice, the bases are related by unimodular transformations: $d \times d$ integral matrices with determinant ± 1.
- You can: 1) permute vectors,
  2) add to a given basis vector another basis vector.
- Interesting bases are made of short and orthogonal vectors.

Lattice volume: $\det(L)$.

- The d-dimensional volume of the parallelepiped spanned by lattice basis vectors, for any basis.
- If $d = n$, absolute value of the determinant of a lattice basis.
- In general, $\det(L) = \sqrt{\det G(\mathbf{b}_1, \ldots, \mathbf{b}_d)}$, where G is the Gram matrix of the $\mathbf{b}_i$'s: $(\langle \mathbf{b}_i, \mathbf{b}_j \rangle)_{i,j}$.
- The orthogonality defect $\frac{\|\mathbf{b}_1\| \ldots \|\mathbf{b}_d\|}{\det(L)}$ gives a measure for the quality of a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_d)$ of L.

The lattice volume is a lattice invariant.

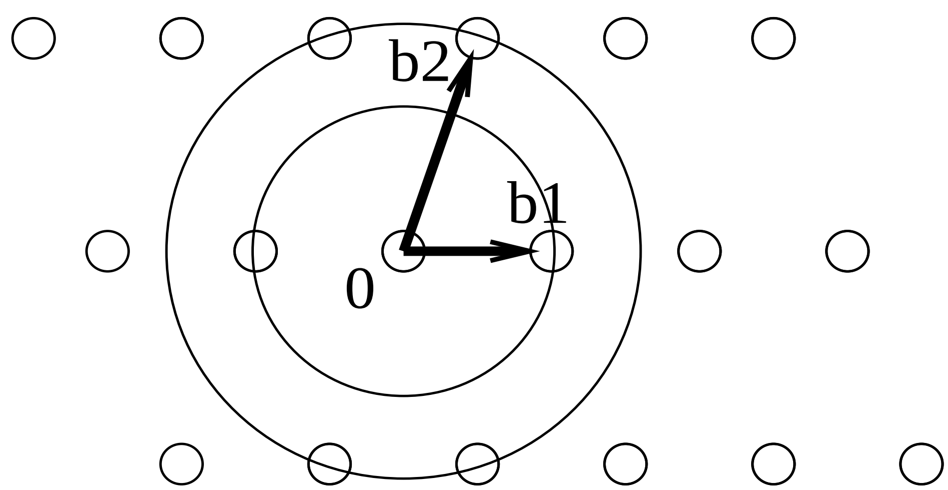


The second definition is not always the good one.

- Suppose that gcd$(a_1, \ldots, a_n) = 1$, and $N \in \mathbb{Z}$.
- $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0 \mod N$.
- The set of solutions $(x_1, \ldots, x_n)$ is a n-dimensional lattice L.
- Let ϕ : $\begin{array}{ccc} \mathbb{Z}^n & \rightarrow & \mathbb{Z} \\ (x_1, \ldots, x_n) & \rightarrow & \sum a_i x_i \mod N \end{array}$.
- $L =$ kerϕ and ϕ is onto $\Rightarrow \mathbb{Z}^n / L \approx \mathbb{Z}_N$.
- $\det(L) = [\mathbb{Z}^n : L] \cdot \det(\mathbb{Z}^n) = N \cdot 1$.

Lattice minima: $\lambda_i(L)$.

- There exists a shortest non-zero vector, its length is $\lambda_1(L)$.
- For $i \leq d$, $\lambda_i(L)$ is the minimum radius r for which $B(\mathbf{0}, r)$ contains i linearly independent lattice vectors.
- Fact: there exist linearly independent vectors reaching the λ_i's.



The two Minkowski theorems.

- Based on the pigeon-hole principle.
- Minkowski 1: $\lambda_1 \leq \sqrt{d} \cdot (\det(L))^{1/d}$.
- Minkowski 2: $\lambda_1 \ldots \lambda_d \leq d^{d/2} \cdot \det(L)$.

- For a "random" lattice, we expect these bounds to be tight:

$$\lambda_1 \approx \lambda_2 \approx \ldots \approx \lambda_d \approx \det(L)^{1/d}.$$

Lattice basis reduction (1/2).

- A reduced basis is made of rather orthogonal and short vectors.
- What would be the best definition?
- A basis reaching the λ_i's? Not always possible when $d \geq 5$:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$\forall i, \lambda_i = 2$, but any basis made of norm-2 vectors is orthogonal.

Lattice basis reduction (2/2).

- Several definitions to work around the failure of the natural one.
- A basis is reduced if the lengths of its vectors are close to the λ_i's.
- Minkowski, Hermite-Korkine-Zolotarev: very strong reductions.
- LLL, BKZ: weaker definitions, but easier to get.

Lattice Related Algorithmic Problems

How to represent a lattice?

- 1st difficulty: a lattice is infinite.

  $\Rightarrow$ A lattice is represented by one of its bases.

- 2nd difficulty: basis vectors may have real coordinates.

  $\Rightarrow$ We consider only integral lattices: sublattices of $\mathbb{Z}^n$.

- Basically, a lattice is represented by a $d \times n$ integral matrix.

The shortest vector problem: SVP.

- Given a basis of L, compute a vector of length $\lambda_1(L)$.
- γ-SVP: Compute a vector of length $\leq \gamma \cdot \lambda_1(L)$.

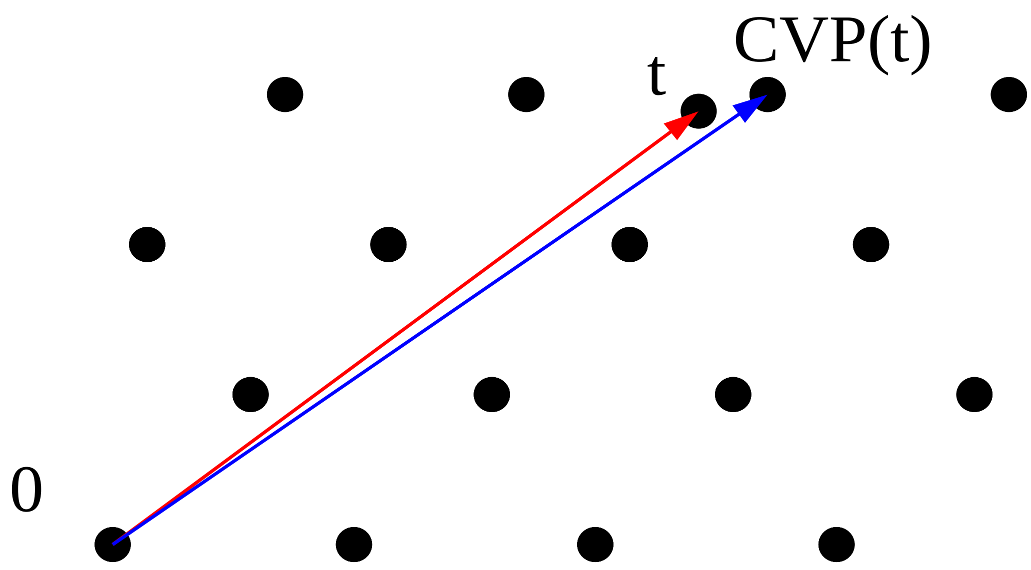
- Expected solution: a vector of length $\approx \det(L)^{1/d}$.
- If λ_1 is much shorter than this, it might be easier.

Effective Minkowski theorem problem: EMTP.

- EMTP: Compute a lattice vector $\mathbf{b}$ with $\|\mathbf{b}\| \leq \sqrt{n} \cdot \det(L)^{1/d}$.
- γ-EMTP: Compute a lattice vector $\mathbf{b}$ with $\|\mathbf{b}\| \leq \gamma \cdot \det(L)^{1/d}$.
- γ-EMTP2: Compute a lattice basis $(\mathbf{b}_1, \ldots, \mathbf{b}_d)$ with $\|\mathbf{b}_1\| \ldots \|\mathbf{b}_d\| \leq \gamma \cdot \det(L)$.

The closest vector problem: CVP.

- Given a basis of L and a vector $\mathbf{t}$ of the embedding space, compute a lattice vector closest to $\mathbf{t}$.
- γ-CVP: Given a basis of L and a target vector $\mathbf{t}$, compute a lattice vector $\mathbf{b}_0$ such that $\|\mathbf{b}_0 - \mathbf{t}\| \leq \gamma \cdot \min_{\mathbf{b} \in L} \|\mathbf{b} - \mathbf{t}\|$.



More on CVP.

- A “general” solution should be at distance $\det(L)^{1/d}$ of $\mathbf{t}$.
- Intuition of the difficulty: Consider $\mathbf{t} = (1/2, \ldots, 1/2)$ and slightly shake $\mathbb{Z}^d$. Which one of the 2^d vertices is the solution?
- CVP is considered harder than SVP.

Lattice Reduction Algorithms

Two different goals.

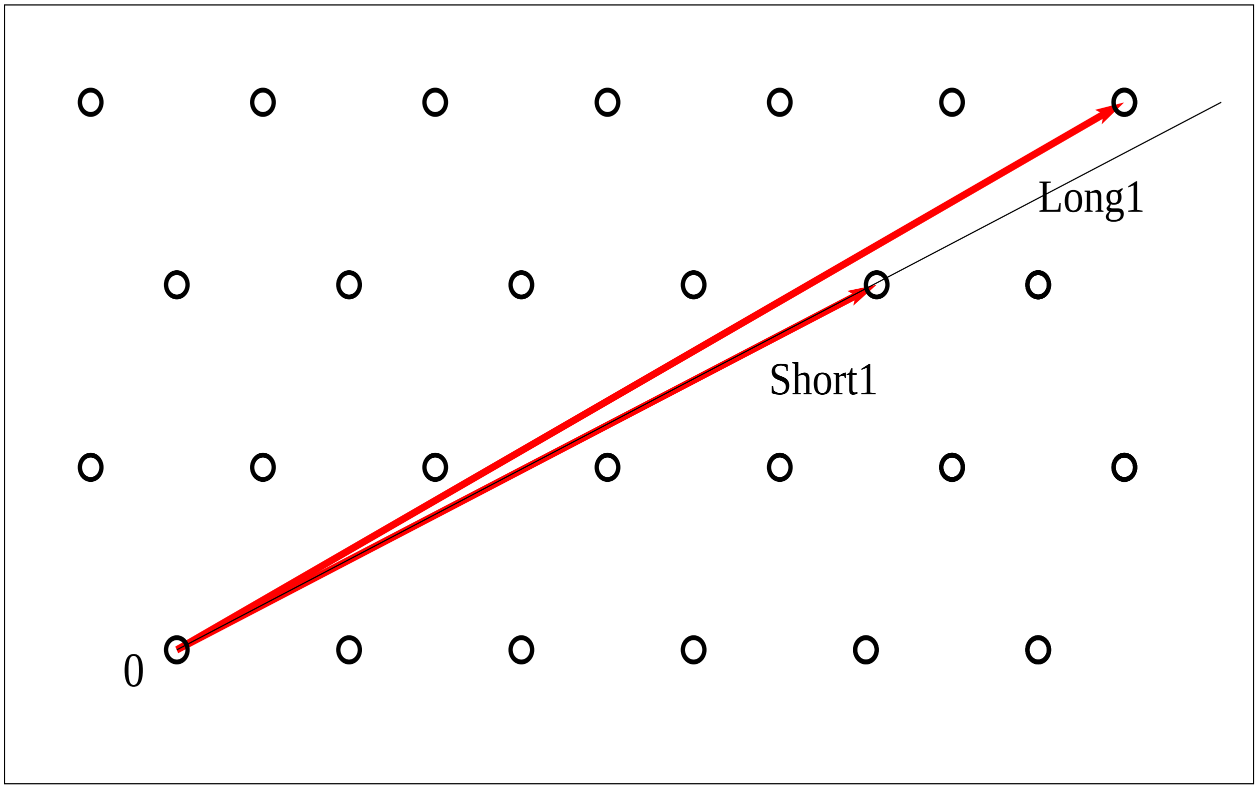
- Get a reasonable approximation factor very quickly.
- Spend some time to get a better approximation factor.

- Often we want a trade-off between both goals.
- What we can afford: Polynomial Time / Practicability.

The 2-dimensional case.

- Gauss (Lagrange?) algorithm solves everything.
- Vectorial generalization of Euclid's algorithm.
- Running time: $O(\log^2 B)$, where $B = \max(\|\mathbf{b}_1^{init}\|, \|\mathbf{b}_2^{init}\|)$.
- Algorithm: shorten the long vector by adding to it an integer multiple of the short one, until this is possible.

The 2-dimensional Case.



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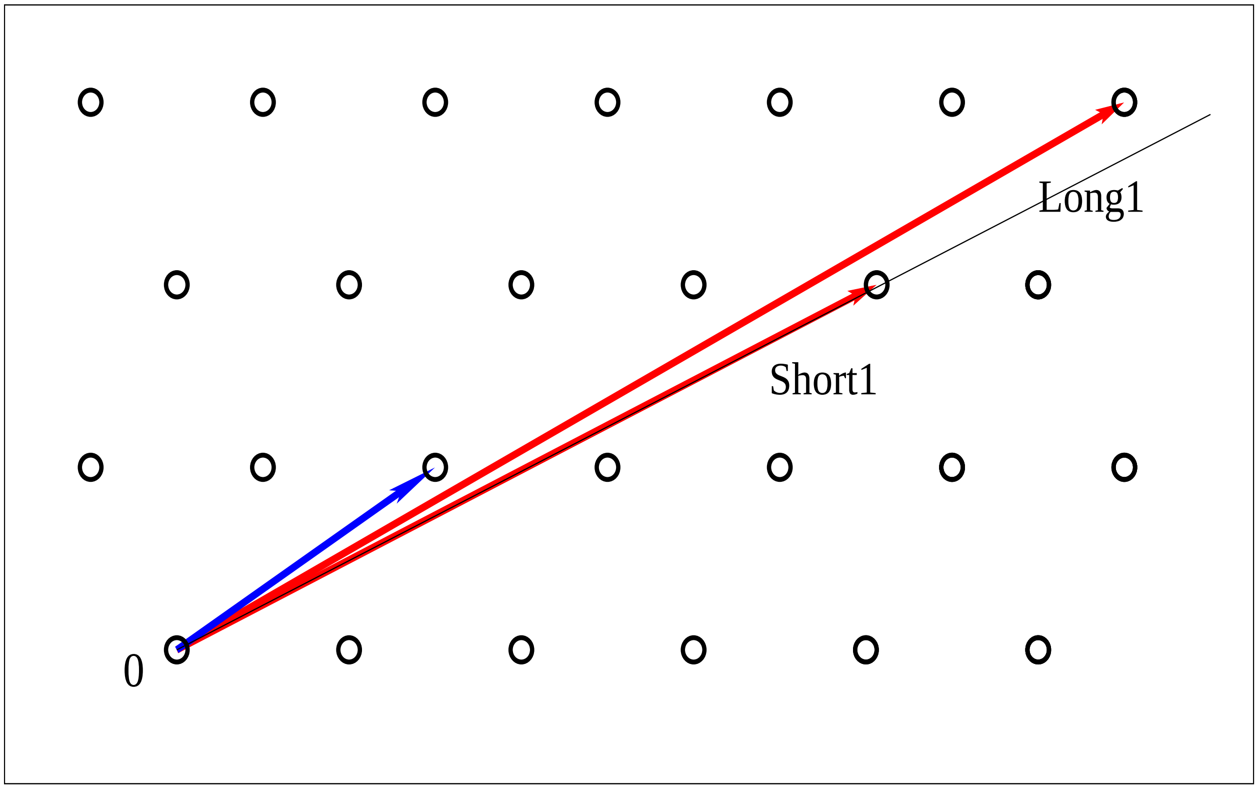

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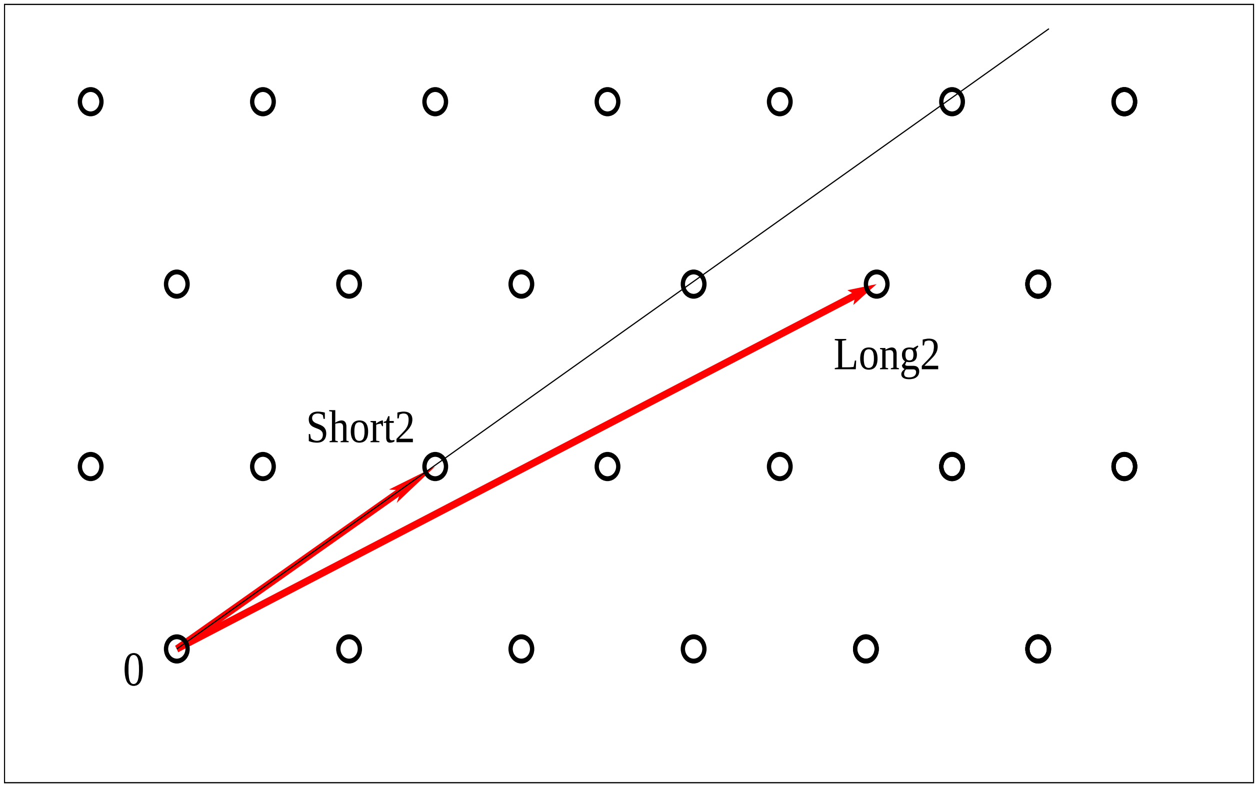

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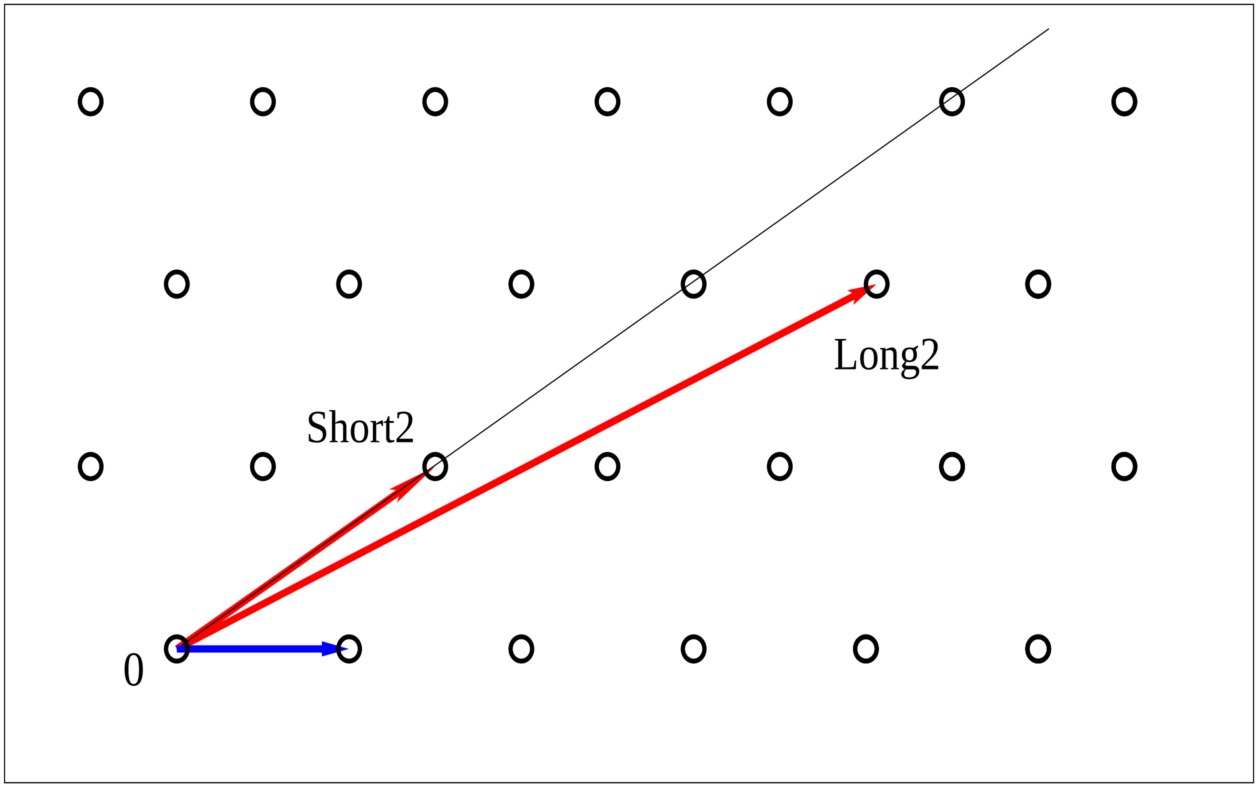

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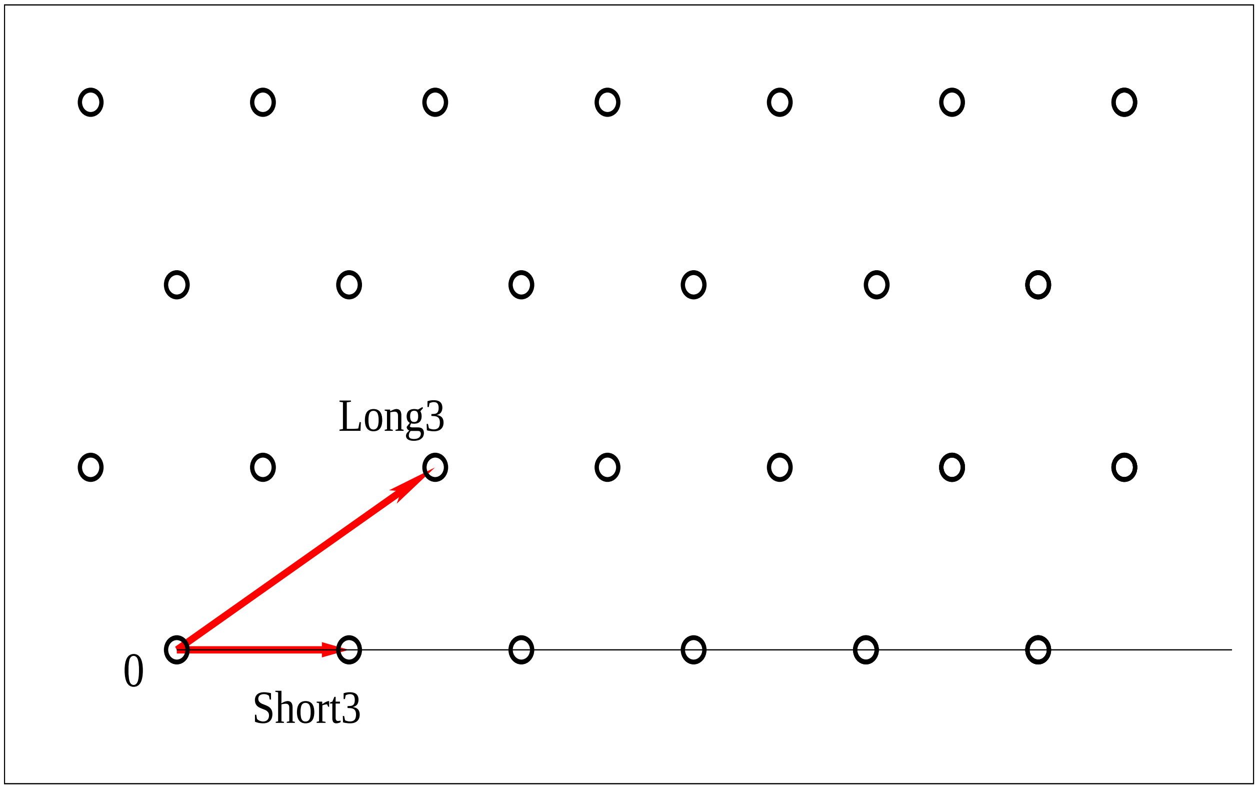

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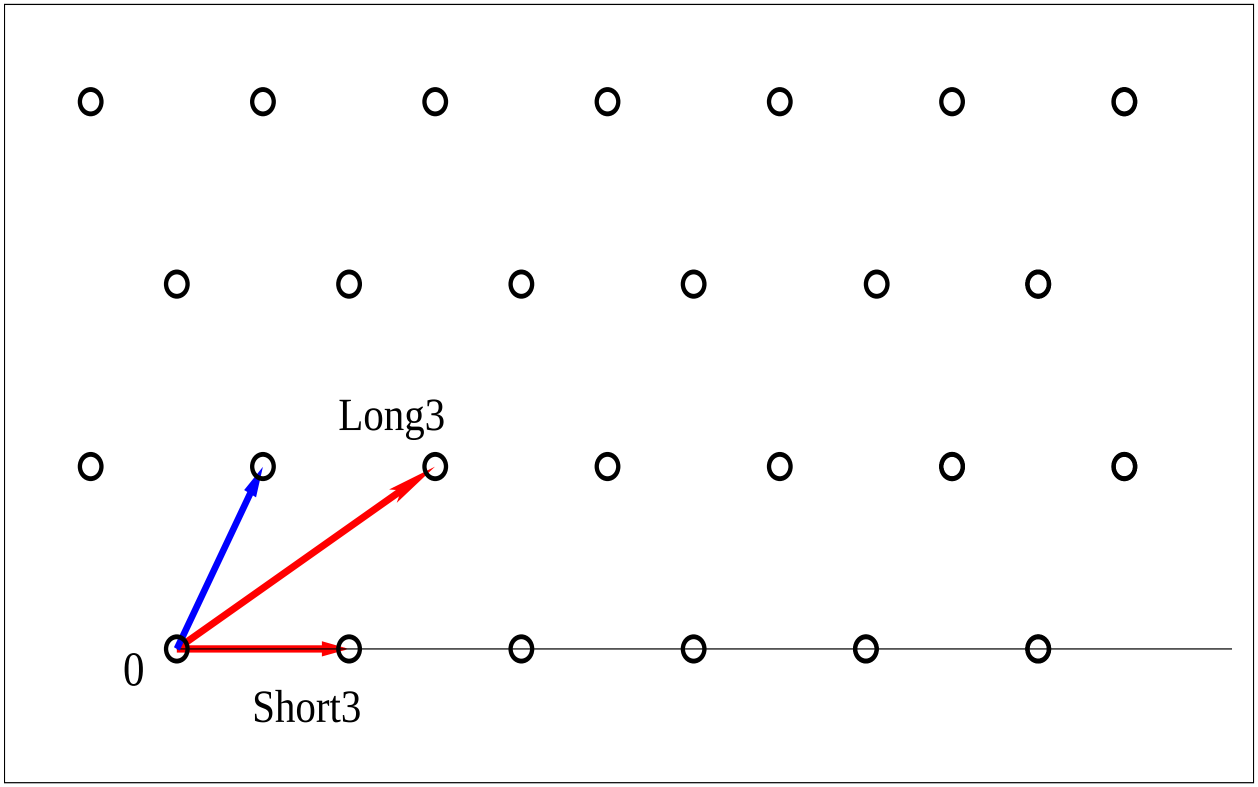

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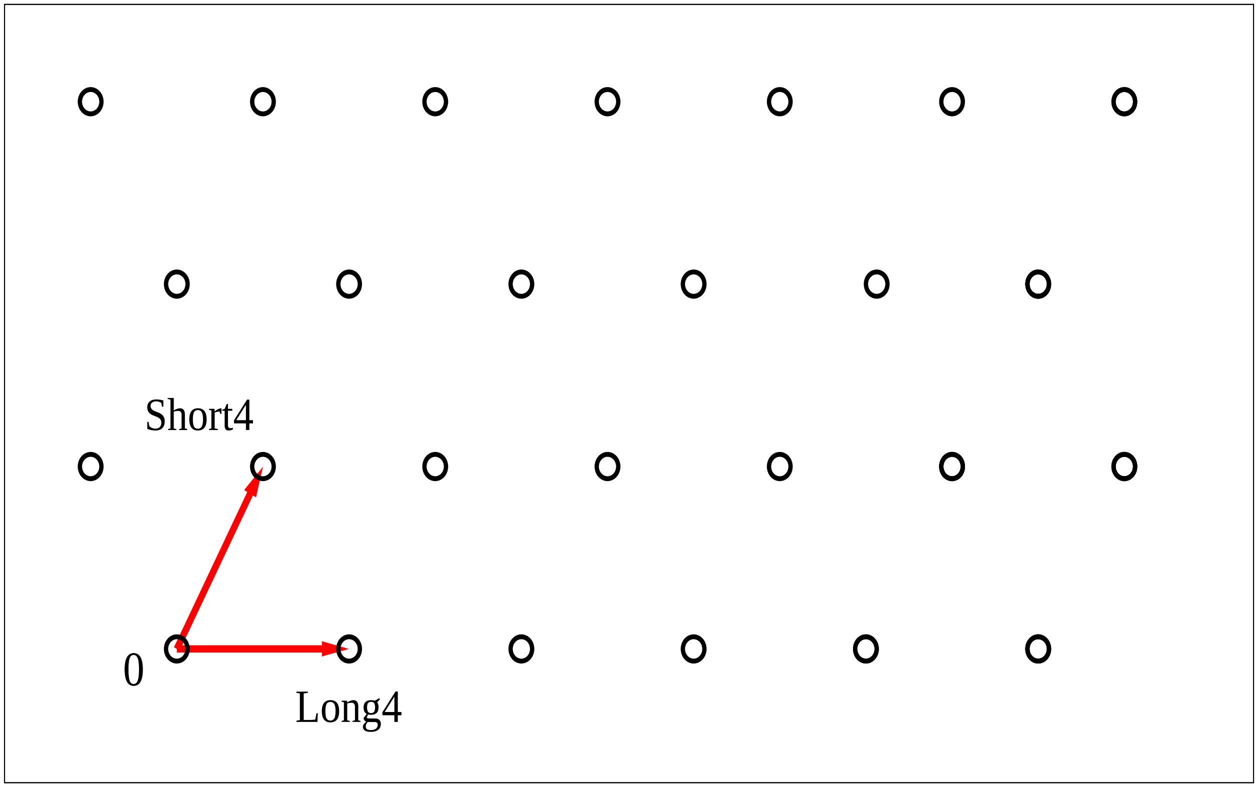

When the dimension remains low.

- Suppose we want a d-dimensional HKZ-reduced basis.
- For small d, exponential algorithms remain feasible.
- Algorithms: Kannan, Ajtai-Kumar-Sivakumar.
- SVP and CVP solved in practice up to dimension $\approx 25 - 30$.

When the dimension grows significantly (1/2).

- Use the LLL algorithm (Lenstra, Lenstra, Lovász - 1982).
- It gives an LLL-reduced basis $(\mathbf{b}_1, \ldots, \mathbf{b}_d)$ with:

$$\begin{aligned} \|\mathbf{b}_1\| &\leq c^d \cdot \det(L)^{1/d}, \\ \|\mathbf{b}_i\| &\leq c^{2d} \cdot \lambda_i(L), \end{aligned}$$

  where $c = (4/3)^{1/4} - \varepsilon \approx 1.075$.

- Time: $O(d^5 n \log^3 B)$, with $B = \max_{i \leq d} \|\mathbf{b}_i^{init}\|$.
- With floating-point arithmetic: $O(d^4 n (d + \log B) \log B)$.

When the dimension grows significantly (2/2).

- What if you want a basis more reduced than given by LLL?
  $\Rightarrow$ Mix LLL and HKZ-reduction.

- This is Schnorr's Block-Korkine-Zolotarev algorithm:
  LLL == BKZ$_2$, HKZ == BKZ$_d$.

- BKZ$_k$ costs $\approx k^{O(k)}$ and gives $\gamma = k^{O(n/k)}$ for SVP.
  $\Rightarrow$ Best γ for deterministic polynomial time: $2^{O\left(k\frac{(\log\log k)^2}{\log k}\right)}$.

- BKZ is feasible for $k \leq 25$ to 30.

Practical Lattice Reduction.

Quoting Shoup's NTL documentation.

"I think it is safe to say that nobody really understands how the LLL algorithm works. The theoretical analyses are a long way from describing what "really" happens in practice. Choosing the best variant for a certain application ultimately is a matter of trial and error."

Lattices arising in real life (1/2).

- Small-dimensional lattices (e.g., in Wiener's attack).
- Knapsack-like lattices (knapsacks):

$$\begin{pmatrix} X_1 & 1 & 0 & \dots & 0 & 0 \\ X_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ X_d & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

  with rather large X_i's, and a large d (100-200).

  Sometimes LLL suffices, but BKZ is usually required.

- Coppersmith-type lattices: very large entries, medium dimension (70), LLL suffices.

Lattices arising in real life (2/2).

NTRU-like lattices: small entries (< 10 bits), very large dimension (167-503), very good reduction is required.

$$\left(\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & h_0 & h_1 & \dots & h_{n-1} \\
0 & 1 & \dots & 0 & h_1 & h_2 & \dots & h_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 1 & h_{n-1} & h_0 & \dots & h_{n-2} \\
\hline
0 & 0 & \dots & 0 & q & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & q & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & q
\end{array}\right).$$

Where can one find lattice algorithms implementations?

- NTL: efficient LLL, improved BKZ.
- Magma: less efficient LLL, nice routines in low dimensions.
- On my webpage: floating-point LLL (no BKZ), quite fast.
- Others: Lidia, Maple, Mathematica, PSLQ.

What can one do in practice?

- Complete (KZ) reduction in low dimensions: $d \leq 25$ to 30.
- LLL-reduction of large lattices ($d \leq 1000$).
  Knapsack-type lattice with $d = 121$ and $\log B = 29000$: 15 min.
  Knapsack-type lattice with $d = 50$ and $\log B = 100000$: 6 min.
- BKZ$_{30}$ in dimension ≤ 300 takes some time but should terminate.

On the quality of bases output by LLL.

Does LLL find vectors much shorter than expected?

- $\|\mathbf{b}\| \leq (4/3)^{d/4} \det(L)^{1/d}$, with $(4/3)^{1/4} \approx 1.075$.
- Experimentally, for "random" lattices: $1.075 \to 1.03$ (?).
- Widespread belief: LLL gives a solution to SVP very often, and approximates SVP very well.
- Explanation: in the 80's, people were working with medium-size lattices, and: $(1.03)^{30} \approx 2.4$, $(1.03)^{50} \approx 4.4$, $(1.03)^{70} \approx 7.9$.
- Yet much remains unknown about its behavior.

Main open problems.

- Comprehension of the practical behavior of LLL and BKZ.
- Faster lattice reduction algorithms.
- An efficient algorithm solving Poly(d)-SVP.

Some bibliography.

- Siegel, Lectures on the Geometry of Numbers.
- Lovász, An Algorithmic Theory of Numbers, Graphs and Convexity.
- Cohen, A Course in Computational Algebraic Number Theory.
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  Complexity of Lattice Problems, A Cryptographic Perspective.