Fully Homomorphic Encryption

Part I

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incl. some slides courtesy of J.-S. Coron

Outline

Homomorphic encryption

- Computing on encrypted data.
- Multiplicatively homomorphic: "textbook RSA".

$$\left.\begin{aligned} c_1 &= m_1^e \bmod N \\ c_2 &= m_2^e \bmod N \end{aligned}\right. \Rightarrow c_1 \cdot c_2 = (m_1 \cdot m_2)^e \bmod N$$

- Additively homomorphic: Paillier.

$$\left.\begin{aligned} c_1 &= g^{m_1} x_1^N \bmod N^2 \\ c_2 &= g^{m_2} x_2^N \bmod N^2 \end{aligned}\right. \Rightarrow c_1 \cdot c_2 = g^{m_1+m_2\ [N]} (x_1 x_2)^N \bmod N^2$$

- Fully homomorphic: homomorphic for both addition and multiplication
  - Open problem until Gentry's breakthrough in 2009.

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Homomorphic encryption for e-voting

- A typical application of additively homomorphic encryption is secure voting schemes.
- In a yes-no election, each voter casts a ballot by encrypting 0 or 1 using the Paillier public key of the organizer of the election.
- The ballots are then shuffled and added together homomorphically by some independent third parties.
- Decrypting the resulting ciphertext reveals the tally, while individual votes remain secret.
- Add in zero-knowledge proofs to ensure that each step is correct, and use threshold encryption/secret sharing to avoid a single authority.
- Make sure voters have Ph.D.'s in cryptography to understand the whole process.

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Timeline of privacy homomorphisms

- [RSA77]: multiplication mod N;
- [RAD78]: introduce the notion of privacy homomorphism
  - almost suggests FHE as an open problem...
- [GM84]: addition mod 2, CPA-security;
- [ElGamal84]: multiplication mod p;
- [Paillier98], [OU98]: addition mod N (resp. mod p);
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Fully homomorphic encryption

- We restrict ourselves to encrypting a single bit:
  - $0 \to 203\text{ef}6124\ldots23\text{ab}87_{16}$
  - $1 \to \text{b}327653\text{c}1\ldots\text{db}3265_{16}$
  - no loss of generality, by the hybrid argument.
- Fully homomorphic property
  - Given $E(b_0)$ and $E(b_1)$, one can compute $E(b_0 \oplus b_1)$ and $E(b_0 \cdot b_1)$ without knowing the private key.
- Computing over a ring:
  - Given a circuit with xors and ands, and encrypted input bits, one can compute the output in encrypted form, without knowing the private key.
  - Hence, compute any function on encrypted data that can be represented as a boolean circuit with polynomially many gates (and $\mathbf{BPP} \subseteq \mathbf{P/poly}$).

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Security notions

- One can consider both secret-key and public-key FHE schemes.
  - Both are interesting.
- Usual security notion: IND-CPA.
  - In the view of an adversary without the secret/private key, $E(0) \cong E(1)$.
- Can we have more?
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Compactness

- Trivial construction of FHE from any encryption scheme:
  - Same key generation and encryption;
  - $\mathsf{Eval}(\mathsf{pk}, f, c) = (c, f)$;
  - $\mathsf{Decrypt}(\mathsf{sk}, (c, f)) = f(\mathsf{Decrypt}(\mathsf{sk}, c))$.
- We want to exclude such trivial constructions, where no computation is actually carried out on ciphertexts.
- Usual extra requirement: compactness.
  - Ciphertext size independent of successive homomorphic operations.
- One can ask for something stronger: circuit privacy.
  - For a given plaintext, ciphertext distribution independent of successive homomorphic operations.
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  - To cast their ballots, voters encrypt $x_i = 0$ or 1 under an additive homomorphic encryption scheme, together with a zero-knowledge proof of equality to 0 or 1;
  - Third parties shuffle the ballots, add the ciphertexts homomorphically, checking all the proofs;
  - Organizers decrypt the tally.
- Using fully homomorphic encryption, do away with the voters' zero-knowledge proofs:
  - In addition to computing the homomorphic sum of the ballots, the third parties can compute a ciphertext for:

$$t = \prod x_i(x_i - 1).$$

  - The organizers can decrypt this ciphertext and check that $t = 0$ to ensure all ballots were valid (equal to 0 or 1).
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What can we do with FHE? (2)

- Possibe business model courtesy of J.-S. Coron:
- You have a software that given the revenue, past income, headcount, etc., of a company can predict its future stock price.
  - I want to know the future stock price of my company, but I don't want to disclose confidential information.
  - And you don't want to give me your software containing secret formulas.
- Using homomorphic encryption:
  - I encrypt all the inputs using fully homomorphic encryption and send them to you in encrypted form.
  - You process all my inputs, viewing your software as a circuit.
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What can we do with FHE? (3)

- Some say FHE is a very nice solution in search of a problem.
- Applications I do not believe in:
  - Fully homomorphic Google queries.
  - Or anything about secure cloud computing, really.
  - Because cloud computing is database search, and doing this with encrypted queries is intrinsically inefficient (linear instead of logarithmic in the size of the database).
  - Unless interactive protocols are fine, but then use PIR.
- Applications that may see the light of day:
  - Handling of data sensitive enough that parties are prepared to pay a heavy price for extra security;
  - involving relatively simple functions (shallow circuits).
- In the meantime, FHE is a powerful crypto primitive that lets you build many advanced protocols (NIZK, MPC, secure databases, etc.) provided you do not care to much about efficiency.

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- 2009: breakthrough scheme by Gentry.
  - Concrete parameters unclear, probably prohibitively inefficient.
- 2010: vDGHV scheme over the integers.
  - Public key size $> 2^{60}$ bits at reasonable security levels!
- 2011: first implementations of these schemes with numerous optimizations [GH11], [CMNT11].
  - 15–30 min. per multiplication gate, public key $\approx$ 1 GB.
- 2011–2013: (R)LWE-based schemes.
  - Simpler, more efficient, more versatile.
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Ingredients for FHE

- Consider an integer lattice $L \subset \mathbb{Z}^n$.
  - Secret/private key: a good basis for the lattice, that can "correct large errors".
  - Public key (optional): a bad basis, or even a noisy bad basis: lets you sample a point close to the lattice, but not distinguish between a point close to the lattice and a random point.
- With this data, we can construct an encryption scheme:
  - $E_{\mathrm{pk}}(0)$ is a point close to the lattice.
  - $E_{\mathrm{pk}}(1)$ is a random point.
  - CPA secure by assumption!
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A more homomorphic construction

- Idea for addition: encode the message in the parity of the noise.
  - Say the pk lets you sample a point of the form $\mathbf{x} + 2\mathbf{e}$ ($\mathbf{x}$ lattice point, $\mathbf{e}$ small random error).
  - Then, encrypt $m \in \{0, 1\}$ as:

$$E_{\mathrm{pk}}(m) = \mathbf{x} + 2\mathbf{e} + (m, 0, \ldots, 0)$$

  - Still CPA, easy to decrypt.
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Gentry's SHE scheme

- Public parameters: n a power of 2, $R = \mathbb{Z}[x]/(x^n + 1)$.
- Key generation returns a lattice L which is an ideal of $\mathbb{Z}[x]/(x^n + 1)$.
  - Private key is a good basis B_{sk} for L, whose fundamental parallelipiped contains the ball of radius d.
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Homomorphic properties of Gentry's scheme

- Addition:

$$\begin{aligned} \mathbf{c}_1 &= \mathbf{x}_1 + 2\mathbf{e}_1 + m_1 \\ \mathbf{c}_2 &= \mathbf{x}_2 + 2\mathbf{e}_2 + m_2 \end{aligned} \Rightarrow \mathbf{c}_1 + \mathbf{c}_2 = \mathbf{x}' + 2\mathbf{e}' + m_1 + m_2$$

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with $\mathbf{e}' = 2\mathbf{e}_1 \cdot \mathbf{e}_2 + m_1\mathbf{e}_2 + m_2\mathbf{e}_1$.

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  - The scheme supports circuits with $\approx \log_2\left(\frac{\log_2 d}{\log_2 \delta}\right)$ levels of Mult gates (somewhat homomorphic encryption).

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Outline

The DGHV Scheme (symmetric version)

- Ciphertext for $m \in \{0, 1\}$:

$$c = q \cdot p + 2r + m$$

where p is the secret key (lattice basis), q and r are randoms.

- Decryption:

$$(c \bmod p) \bmod 2 = m$$

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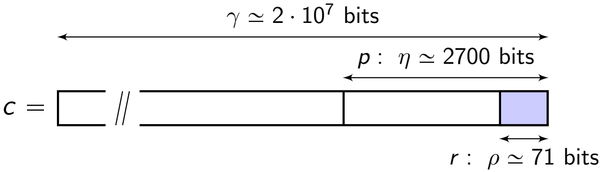

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Public-key encryption with vDGHV

- We need to provide a "noisy" description of the ideal $p\mathbb{Z}$
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- As described here, reduces to the General Approximate GCD (GACD) problem: given polynomially many close multiples of p, find p.
  - Idea of the reduction: using an adversary that distinguishes $E(0)$ and $E(1)$ with significant probability, construct an algorithm that predicts the LSB of q in $q \cdot p + r$ with high probability. Conclude using binary GCD.
- In practice, we change the algorithm slightly, by adding an exact multiple of p, $x_0 = q_0 \cdot p$, in the public key.
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- The number of multiplications is limited.
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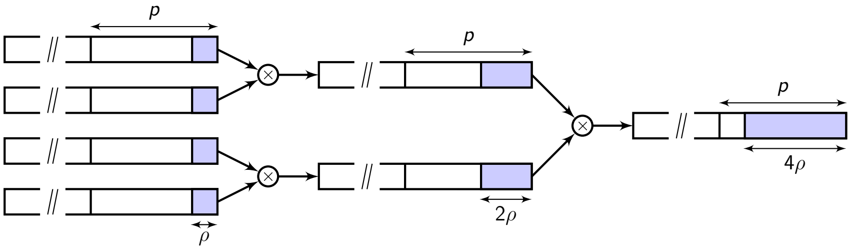
- This is a problem with all schemes in this framework.



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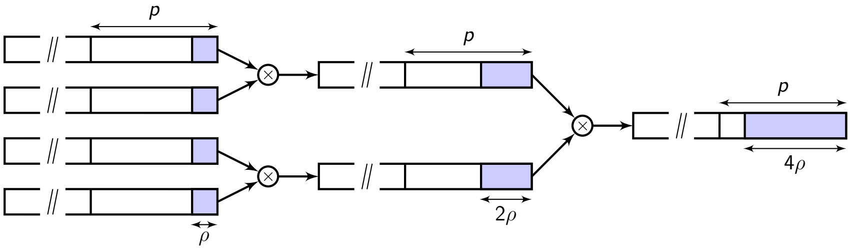
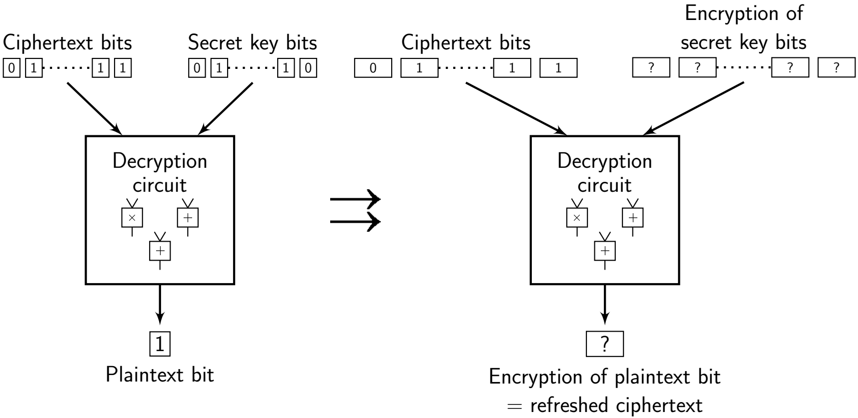


Solution: Bootstrapping

- Gentry's breakthrough idea: refresh the ciphertext by evaluating the decryption circuit homomorphically: bootstrapping.



Ciphertext refresh

- Refreshed ciphertext:
  - If the degree of the decryption polynomial is small enough, the resulting noise in this new ciphertext can be smaller than in the original ciphertext
- Fully homomorphic encryption:
  - Given two refreshed ciphertexts one can apply again the homomorphic operation (either addition or multiplication), which was not necessarily possible on the original ciphertexts because of the noise threshold.
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- Do we know that the encryption scheme remains secure even after publishing encryption of the secret key bits?
  - This is called circular security.
  - Only a couple of encryption schemes are proved circular secure, none of them fully homomorphic.
  - Add circular security as an ad hoc assumption.
- The noise of refreshed ciphertexts depends on the AND-depth d of the decryption circuit (it is roughly $d\rho$, where ρ is the noise of fresh ciphertexts).
  - But d can be huge! In vDGHV, it is the depth of the circuit computing $(c \bmod p) \bmod 2$ given c and the bits of p.
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The squashed vDGHV scheme (idea)

- Write decryption as:

$$m \leftarrow [c]_2 \oplus [\lfloor c \cdot (1/p) \rceil]_2$$

This formula can be used for ciphertext refresh if $1/p$ can be put in a compact encrypted form in the public key.

- Idea (Gentry): use secret sharing. Represent $1/p$ as a sparse subset sum:

$$\lfloor 2^\kappa / p \rceil = \sum_{i=1}^{\Theta} s_i \cdot u_i$$

with random κ-bit integers u_i, and $s_i \in \{0,1\}$. Publish the u_i's and encryptions of the s_i's.

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