Fully Homomorphic Encryption
Part I

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incl. some slides courtesy of J.-S. Coron
Introduction

Fully homomorphic encryption
FHE in practice?

Gentry’s original framework for FHE
Gentry’s scheme
The vDGHV scheme
Bootstrapping
Homomorphic encryption

- Computing on encrypted data.
  - Multiplicatively homomorphic: “textbook RSA”.
    \[ c_1 = m_1^e \mod N \]
    \[ c_2 = m_2^e \mod N \]
    \[ \Rightarrow c_1 \cdot c_2 = (m_1 \cdot m_2)^e \mod N \]
  - Additively homomorphic: Paillier.
    \[ c_1 = g^{m_1} x_1^N \mod N^2 \]
    \[ c_2 = g^{m_2} x_2^N \mod N^2 \]
    \[ \Rightarrow c_1 \cdot c_2 = g^{m_1 + m_2} [N](x_1 x_2)^N \mod N^2 \]
  - Fully homomorphic: homomorphic for both addition and multiplication
    - Open problem until Gentry’s breakthrough in 2009.
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Homomorphic encryption for e-voting

- A typical application of additively homomorphic encryption is secure voting schemes.
  - In a yes-no election, each voter casts a ballot by encrypting 0 or 1 using the Paillier public key of the organizer of the election.
  - The ballots are then shuffled and added together homomorphically by some independent third parties.
  - Decrypting the resulting ciphertext reveals the tally, while individual votes remain secret.
  - Add in zero-knowledge proofs to ensure that each step is correct, and use threshold encryption/secret sharing to avoid a single authority.
  - Make sure voters have Ph.D.’s in cryptography to understand the whole process.
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Timeline of privacy homomorphisms

- [RSA77]: multiplication mod $N$;
- [RAD78]: introduce the notion of privacy homomorphism almost suggests FHE as an open problem...
- [GM84]: addition mod 2, CPA-security;
- [ElGamal84]: multiplication mod $p$;
- [Paillier98], [OU98]: addition mod $N$ (resp. mod $p$);
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Fully homomorphic encryption

- We restrict ourselves to encrypting a single bit:
  - $0 \rightarrow 203ef6124 \ldots 23ab87_{16}$
  - $1 \rightarrow b327653c1 \ldots db3265_{16}$
  - no loss of generality, by the hybrid argument.

- Fully homomorphic property
  - Given $E(b_0)$ and $E(b_1)$, one can compute $E(b_0 \oplus b_1)$ and $E(b_0 \cdot b_1)$ without knowing the private key.

- Computing over a ring:
  - Given a circuit with xors and ands, and encrypted input bits, one can compute the output in encrypted form, without knowing the private key.
  - Hence, compute any function on encrypted data that can be represented as a boolean circuit with polynomially many gates (and $\text{BPP} \subseteq \text{P/poly}$).
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Security notions

- One can consider both secret-key and public-key FHE schemes.
  - Both are interesting.
- Usual security notion: IND-CPA.
  - In the view of an adversary without the secret/private key, $E(0) \cong E(1)$.
- Can we have more?
  - CCA2 is incompatible with homomorphic properties.
  - CCA1 is possible, some conversions exist.
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Compactness

- Trivial construction of FHE from any encryption scheme:
  - Same key generation and encryption;
  - Eval(pk, f, c) = (c, f);
  - Decrypt(sk, (c, f)) = f(Decrypt(sk, c)).

- We want to exclude such trivial constructions, where no computation is actually carried out on ciphertexts.
- Usual extra requirement: compactness.
  - Ciphertext size independent of successive homomorphic operations.
- One can ask for something stronger: circuit privacy.
  - For a given plaintext, ciphertext distribution independent of successive homomorphic operations.
  - Rather costly to achieve; usually relaxed somewhat in “practical” schemes.
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What can we do with FHE? (1)

- Recall the secure voting protocol from a few slides back:
  - To cast their ballots, voters encrypt $x_i = 0$ or $1$ under an additive homomorphic encryption scheme, together with a zero-knowledge proof of equality to 0 or 1;
  - Third parties shuffle the ballots, add the ciphertexts homomorphically, checking all the proofs;
  - Organizers decrypt the tally.

- Using fully homomorphic encryption, do away with the voters’ zero-knowledge proofs:
  - In addition to computing the homomorphic sum of the ballots, the third parties can compute a ciphertext for:
    
    $$ t = \prod x_i(x_i - 1). $$

  - The organizers can decrypt this ciphertext and check that $t = 0$ to ensure all ballots were valid (equal to 0 or 1).

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Possibe business model courtesy of J.-S. Coron:

- You have a software that given the revenue, past income, headcount, etc., of a company can predict its future stock price.
  - I want to know the future stock price of my company, but I don’t want to disclose confidential information.
  - And you don’t want to give me your software containing secret formulas.

- Using homomorphic encryption:
  - I encrypt all the inputs using fully homomorphic encryption and send them to you in encrypted form.
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What can we do with FHE? (3)

- Some say FHE is a very nice solution in search of a problem.
  - Applications I do not believe in:
    - Fully homomorphic Google queries.
    - Or anything about secure cloud computing, really.
    - Because cloud computing is database search, and doing this with encrypted queries is intrinsically inefficient (linear instead of logarithmic in the size of the database).
    - Unless interactive protocols are fine, but then use PIR.
  - Applications that may see the light of day:
    - Handling of data sensitive enough that parties are prepared to pay a heavy price for extra security;
    - involving relatively simple functions (shallow circuits).
  - In the meantime, FHE is a powerful crypto primitive that lets you build many advanced protocols (NIZK, MPC, secure databases, etc.) provided you do not care to much about efficiency.
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Efficiency is improving

- 2009: breakthrough scheme by Gentry.
  - Concrete parameters unclear, probably prohibitively inefficient.
- 2010: vDGHV scheme over the integers.
  - Public key size $> 2^{60}$ bits at reasonable security levels!
- 2011: first implementations of these schemes with numerous optimizations [GH11], [CMNT11].
  - 15–30 min. per multiplication gate, public key $\approx 1$ GB.
- 2011–2013: (R)LWE-based schemes.
  - Simpler, more efficient, more versatile.
  - Optimized implementations evaluate the full AES circuit in 10–30 min. amortized.
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- Consider an integer lattice $L \subseteq \mathbb{Z}^n$.
  - Secret/private key: a good basis for the lattice, that can "correct large errors".
  - Public key (optional): a bad basis, or even a noisy bad basis: lets you sample a point close to the lattice, but not distinguish between a point close to the lattice and a random point.

- With this data, we can construct an encryption scheme:
  - $E_{pk}(0)$ is a point close to the lattice.
  - $E_{pk}(1)$ is a random point.
  - CPA secure by assumption!
  - Almost additively homomorphic.
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A more homomorphic construction

- Idea for addition: encode the message in the parity of the noise.
  - Say the pk lets you sample a point of the form $x + 2e$ ($x$ lattice point, $e$ small random error).
  - Then, encrypt $m \in \{0, 1\}$ as:
    \[ E_{pk}(m) = x + 2e + (m, 0, \ldots, 0) \]
  - Still CPA, easy to decrypt.
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- For multiplication, use an ideal lattice.
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Gentry’s SHE scheme

- Public parameters: \( n \) a power of 2, \( R = \mathbb{Z}[x]/(x^n + 1) \).
  - Key generation returns a lattice \( L \) which is an ideal of \( \mathbb{Z}[x]/(x^n + 1) \).
    - Private key is a good basis \( B_{sk} \) for \( L \), whose fundamental parallelipiped contains the ball of radius \( d \).
    - Public key is a bad basis \( B_{pk} \) for \( L \) (usually the HNF); with it, decisional BDD up to distance \( d \) should be hard.
- Encrypt \( \text{Encrypt}(pk, m) = 2^e + m \mod B_{pk} \), with \( e \) random such that \( \|e\| < \delta \).
  - Thus a ciphertext is of the form \( x + 2^e + m \) for some \( x \in L \).
- Decrypt \( \text{Decrypt}(sk, c) = (c \mod B_{sk}) \mod 2 \).
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Homomorphic properties of Gentry’s scheme

- Addition:

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\begin{align*}
\mathbf{c}_1 &= x_1 + 2e_1 + m_1 \\
\mathbf{c}_2 &= x_2 + 2e_2 + m_2
\end{align*}
\Rightarrow \mathbf{c}_1 + \mathbf{c}_2 = \mathbf{x'} + 2\mathbf{e'} + m_1 + m_2
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with \( \mathbf{e}' = 2\mathbf{e}_1 \cdot \mathbf{e}_2 + m_1 \mathbf{e}_2 + m_2 \mathbf{e}_1 \).

- In particular, \( \|\mathbf{e}'\| \lesssim 2\sqrt{n}\|\mathbf{e}_1\| \cdot \|\mathbf{e}_2\| \).

- The scheme supports circuits with \( \approx \log_2 \left( \frac{\log_2 d}{\log_2 \delta} \right) \) levels of Mult gates (somewhat homomorphic encryption).
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The DGHV Scheme (symmetric version)

- Ciphertext for $m \in \{0, 1\}$:
  \[
  c = q \cdot p + 2r + m
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  where $p$ is the secret key (lattice basis), $q$ and $r$ are randoms.

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- Parameters:
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  \gamma \simeq 2 \cdot 10^7 \text{ bits}
  
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- We need to provide a “noisy” description of the ideal $p\mathbb{Z}$

- Ciphertext

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- Public-key: a set of $\tau$ encryptions of 0’s.

\[ x_i = q_i \cdot p + 2r_i \]

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Security of the scheme

- As described here, reduces to the General Approximate GCD (GACD) problem: given polynomially many close multiples of $p$, find $p$.
  - Idea of the reduction: using an adversary that distinguishes $E(0)$ and $E(1)$ with significant probability, construct an algorithm that predicts the LSB of $q$ in $q \cdot p + r$ with high probability. Conclude using binary GCD.

- In practice, we change the algorithm slightly, by adding an exact multiple of $p$, $x_0 = q_0 \cdot p$, in the public key.
  - Then, homomorphic addition an multiplication can be done mod $x_0$, keeping ciphertexts from growing exponentially.
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Solution: Bootstrapping

- Gentry’s breakthrough idea: refresh the ciphertext by evaluating the decryption circuit homomorphically: bootstrapping.

![Diagram showing the process of bootstrapping in fully homomorphic encryption (FHE)].

- Ciphertext bits: 0 1 ··· 1 1
- Secret key bits: 0 1 ··· 1 0
- Decryption circuit:
  - 1
- Plaintext bit
- Encryption of secret key bits:
  - ? ? ··· ? ?
- Ciphertext bits:
  - 0 1 ··· 1 1
- Encryption of plaintext bit:
  - ?

Encryption of plaintext bit = refreshed ciphertext
Ciphertext refresh

- Refreshed ciphertext:
  - If the degree of the decryption polynomial is small enough, the resulting noise in this new ciphertext can be smaller than in the original ciphertext.

- Fully homomorphic encryption:
  - Given two refreshed ciphertexts one can apply again the homomorphic operation (either addition or multiplication), which was not necessarily possible on the original ciphertexts because of the noise threshold.
  - Using this ciphertext refresh (or recryption) procedure, the number of homomorphic operations becomes unlimited and we get a fully homomorphic encryption scheme.
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Problems with bootstrapping

- Do we know that the encryption scheme remains secure even after publishing encryption of the secret key bits?
  - This is called circular security.
  - Only a couple of encryption schemes are proved circular secure, none of them fully homomorphic.
  - Add circular security as an ad hoc assumption.

- The noise of refreshed ciphertexts depends on the AND-depth $d$ of the decryption circuit (it is roughly $d\rho$, where $\rho$ is the noise of fresh ciphertexts).
  - But $d$ can be huge! In vDGHV, it is the depth of the circuit computing $(c \mod p) \mod 2$ given $c$ and the bits of $p$.
  - Probably impossible to set parameters making the scheme bootstrappable as is.
  - We need squashing: change the decryption algorithm to make it low depth. Quite technical.
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The squashed vDGHV scheme (idea)

- Write decryption as:

\[ m \leftarrow [c]_2 \oplus [[c \cdot (1/p)]]_2 \]

This formula can be used for ciphertext refresh if \( 1/p \) can be put in a compact encrypted form in the public key.

- Idea (Gentry): use secret sharing. Represent \( 1/p \) as a sparse subset sum:

\[
\frac{2^{\kappa}}{p} = \sum_{i=1}^{\Theta} s_i \cdot u_i
\]

with random \( \kappa \)-bit integers \( u_i \), and \( s_i \in \{0, 1\} \). Publish the \( u_i \)'s and encryptions of the \( s_i \)'s.

- The decryption function can then be expressed as a polynomial of low degree (30) in the \( s_i \)'s.
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A little game

- In 2012, the best paper award of a minor Indian conference went to an “improvement” of the vDGHV scheme that goes basically like this:
  - Only two public key elements $x_0 = q_0 \cdot p$, $x_1 = q_1 \cdot p + 2r_1$.
  - Encrypt $m$ as $c = m + 2r'_0 + r'_1x_1 \mod x_0$ for small random $r'_0$, $r'_1$.
  - Decrypt as before.

- Game for tomorrow:
  - Show that one can decrypt any ciphertext with the public key alone!
  - Hint: this involves lattice reduction in very small dimension.
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