Comet, exercises 6
Bring your answers to next course (Oct 26)

1 Universal Coalgebra

Let \( F = 2 \times Id^A \) (i.e., \( FX = 2 \times X^A \)) be the functor for deterministic automata.

**Question 1.1.** Give its action on morphisms (i.e., what is \( Ff \) for some \( f : X \to Y \) ?). Prove that it is a functor.

**Answer.** \( Ff \) is the following function from \( FX = 2 \times X^A \) to \( FY = 2 \times Y^A \):
\[
Ff : 2 \times X^A \to 2 \times Y^A
\]
\[
\langle o, t \rangle \mapsto \langle o, f \circ t \rangle
\]

Given this definition, that \( Fid = id \) and \( F(f \circ g) = Ff \circ Fg \) is obvious. \( \square \)

Recall that the final coalgebra for this functor is the coalgebra of formal languages on the alphabet \( A \), with derivatives describing the dynamics:
\[
\langle \mathcal{P}(A^*), \langle \epsilon, \delta \rangle \rangle \text{ with } \epsilon(L) = " \epsilon \in L " \text{ and } \delta_a(L) = a^{-1}L = \{ w \mid aw \in L \}.
\]

**Question 1.2.** Describe the final coalgebra for the functors \( B \times Id^A \) and \( B \times Id \), where \( B \) is an arbitrary set (justify your answers).

**Answer.** For \( B \times Id^A \), the final algebra consists of the set \( B^{A^*} \) of functions from finite words on \( A \), to \( B \). (The case \( B = 2 \) gives back formal languages, represented by their characteristic function.)

The coalgebra structure is given as follows:
\[
z : B^{A^*} \to B \times (B^{A^*})^A
\]
\[
f \mapsto [f(\epsilon), (a \mapsto w \mapsto f(aw))]
\]

Given a coalgebra \( f : X \to B \times X^A \), one defines the following function \([\cdot] : X \to B^{A^*} \) by induction on words:
\[
[x](\epsilon) = \pi_1(f(x))
\]
\[
[x](aw) = ([\pi_2(f(x))](a))[w]
\]

(Writing \( f = \langle o, t \rangle \) with \( o = \pi_1 \circ f \) and \( t = \pi_2 \circ f \), we get the more friendly notations \( [x](\epsilon) = o(x) \) and \( [x](aw) = [t(x)(a)](w) \).

I let you check that this is the unique function such that \( z \circ [\cdot] = F[\cdot] \circ f \) (i.e., the unique coalgebra homomorphism from \( \langle X, f \rangle \) to \( \langle B^{A^*}, z \rangle \).)
For the functor $B \times \text{Id}$, just apply the previous answer to $A = 1$ (any singleton set). The final coalgebra is thus $B^1$, but $1^\ast \simeq \mathbb{N}$ so that we get functions from natural numbers to $B$, i.e., infinite streams of elements of $B$. \qed