Comet – excercise 7

Bring your answers to the course on Nov. 17th.

1 Up-to techniques for weighted automata

Recall the following monotone maps on the lattice $\text{Rel}_V$ of relations over a semimodule $V$ for a semiring $k$.

- $u(R) = \{(v, w) \mid v = v_1 + v_2, w = w_1 + w_2, (v_1, w_1) \in R, (v_2, w_2) \in R\}$
- $\cdot(R) = \{(k \cdot v, k \cdot w) \mid (v, w) \in R, k \in k\}$
- $id(R) = R$
- $r(R) = \{(v, v) \mid v \in V\}$
- $s(R) = \{(v, w) \mid (w, v) \in R\}$
- $t(R) = \{(v, w) \mid \exists u, (v, u) \in R, (u, w) \in R\}$

The congruence closure is defined as expected

$$ c = (id \cup r \cup s \cup t \cup \cdot)^\omega $$

and the contextual closure as

$$ ctx = (id \cup r \cup u \cup \cdot)^\omega. $$

**Question 1.1.** Prove that, whenever $k$ is a field, every bisimulation up-to $c$ is a bisimulation up-to $ctx$.

**Question 1.2** (Optional). Provide an example of bisimulation up-to $c$ that is not a bisimulation up-to $ctx$. [Hint: use one of the semiring seen during the course]

2 Streams of natural numbers

Weighted automata over a singleton alphabet $A = \{\bullet\}$ and the semiring of natural numbers $N$ accept streams (namely infinite sequences) of natural numbers. During the course we have already seen the weighted automata accepting the Fibonacci’s serie $0, 1, 1, 2, 3, 5, 8 \ldots$

**Question 2.1.** Give a weighted automata accepting the serie $1, 2, 4, 8, \ldots$

**Question 2.2.** Give a weighted automata accepting the serie $1, 1, 1, 1, \ldots$. One for the serie $1, 2, 3, 4 \ldots$. One for the serie $1^2, 2^2, 3^2, 4^2 \ldots$ and one for $1^3, 2^3, 3^3, 4^3 \ldots$

**Question 2.3** (Optional). All the series of the previous question have the shape $1, 2^n, 3^n, 4^n \ldots$. Can you provide the general rule for the weighted automata accepting such serie for an arbitrary $n$?

**Question 2.4** (Optional). Not all the series of natural numbers can be accepted by a finite weighted automaton. What about the serie of Catalan numbers $1, 1, 2, 5, 14, 42, \ldots$? You do not need to provide a formal proof.

During the course we have seen that whenever the underlying semiring is a field, the problem of deciding the equivalence of finite weighted automata is decidable.

**Question 2.5.** Show how to reduce the problem of equivalence of weighted automata over a singleton alphabet and the semiring of natural numbers to the above problem.