Comet, exercises 8 and 9

Bring your answers to next course (Nov 16)

1 Weighted Automata

Consider a weighted automaton \((S, o, t)\) over the input alphabet \(A\) with weights on a semiring \(\mathbb{R}\). Recall that the maps \(o^\#: \mathbb{R}_S^ω \to \mathbb{R}\) and \(t^\#: \mathbb{R}_S^ω \to (\mathbb{R}_S^ω)^A\) – defined during the course – are semi-module homomorphisms.

Let \(\text{Rel}_{\mathbb{R}_S^ω}\) be the complete lattice of relations over \(\mathbb{R}_S^ω\). The monotone map \(b: \text{Rel}_{\mathbb{R}_S^ω} \to \text{Rel}_{\mathbb{R}_S^ω}\) is defined for all relations \(R \in \text{Rel}_{\mathbb{R}_S^ω}\) as

\[
\begin{aligned}
(b(R)) &= \{(v_1, v_2) \text{ s.t. } o^\#(v_1) = o^\#(v_2) \text{ and for all } a \in A, (t^\#_a(v_1), t^\#_a(v_2)) \in R\}.
\end{aligned}
\]

Consider the monotone map \(c: \text{Rel}_{\mathbb{R}_S^ω} \to \text{Rel}_{\mathbb{R}_S^ω}\) defined as \((\text{id} \cup r \cup s \cup t \cup u \cup \bullet)^\#,\) where \(r,s,t\) are defined as usual (see e.g. the DM) and \(u\) and \(\bullet\) are defined for all relations \(R \in \text{Rel}_{\mathbb{R}_S^ω}\) as

\[
\begin{aligned}
u(R) &= \{(v_1, v_2) \text{ s.t. } v_1 = v_1' + v_1'' \text{ and } v_2 = v_2' + v_2'' \text{ and } (v_1', v_2') \in R, (v_1'', v_2'') \in R\},
\end{aligned}
\]

\[
\begin{aligned}
\bullet(R) &= \{(v_1, v_2) \text{ s.t. } v_1 = k \cdot v_1', v_2 = k \cdot v_2' \text{ and } (v_1', v_2') \in R\}.
\end{aligned}
\]

Question 1.1 Prove that \(c\) is sound with respect to \(b\).

Question 1.2 Suppose to have an oracle to decide whether \((v_1, v_2) \in c(R)\) for all vectors \(v_1,v_2 \in \mathbb{R}_S^ω\) and relations \(R \in \text{Rel}_{\mathbb{R}_S^ω}\). Consider the algorithm HKC for weighted automata seen at lesson. Prove that if HKC\((v_1, v_2)\) returns true, then \(v_1\) and \(v_2\) accepts the same weighted language.

2 Streams

Let \(\mathbb{R}\) be the field of real numbers. Consider the following stream GSOS specification on \(\mathbb{R}\).

\[
\begin{array}{c c c c c c}
\hline
n \rightarrow & x \rightarrow & y \rightarrow & x' \rightarrow & y' \rightarrow & n+m \rightarrow \oplus \rightarrow \oplus \\
\hline
\end{array}
\]

\[
\begin{array}{c c c c c c}
\hline
x \rightarrow & y \rightarrow & x' \oplus y' \rightarrow & (n \oplus y') \rightarrow & (x' \oplus y) \\
\hline
\end{array}
\]

Question 2.1 Use coinduction to prove associativity and commutativity of \(\oplus\), that is for all streams \(\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^ω\),

\[
(\sigma_1 \oplus \sigma_2) \oplus \sigma_3 = \sigma_1 \oplus (\sigma_2 \oplus \sigma_3) \quad \text{and} \quad \sigma_1 \oplus \sigma_2 = \sigma_2 \oplus \sigma_1.
\]

Question 2.2 Use coinduction up-to and the previous result to prove distributivity of \(\otimes\) over \(\oplus\), that is for all streams \(\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^ω\),

\[
\sigma_1 \otimes (\sigma_2 \oplus \sigma_3) = (\sigma_1 \otimes \sigma_2) \oplus (\sigma_1 \otimes \sigma_3).
\]