Comet, exercises 8 and 9

Bring your answers to next course (Nov 16)

1 Weighted Automata

Consider a weighted automaton (S, o, t) over the input alphabet A with weights on a semiring $\mathbb{R}$. Recall that the maps $o^\sharp \colon \mathbb{R}^S_\omega \to \mathbb{R}$ and $t^\sharp \colon \mathbb{R}^S_\omega \to (\mathbb{R}^S_\omega)^A$ – defined during the course – are *semi-module homomorphisms.*

Let $\mathbf{Rel}_{\mathbb{R}^S_\omega}$ be the complete lattice of relations over $\mathbb{R}^S_\omega$. The monotone map $b \colon \mathbf{Rel}_{\mathbb{R}^S_\omega} \to \mathbf{Rel}_{\mathbb{R}^S_\omega}$ is defined for all relations $R \in \mathbf{Rel}_{\mathbb{R}^S_\omega}$ as

$$b(R) = \{(v_1, v_2) \text{ s.t. } o^\sharp(v_1) = o^\sharp(v_2) \text{ and for all } a \in A,\ (t^\sharp_a(v_1), t^\sharp_a(v_2)) \in R\}.$$

Consider the monotone map $c \colon \mathbf{Rel}_{\mathbb{R}^S_\omega} \to \mathbf{Rel}_{\mathbb{R}^S_\omega}$ defined as $(id \cup r \cup s \cup t \cup u \cup \bullet)^\omega$, where r, s, t are defined as usual (see e.g. the DM) and u and $\bullet$ are defined for all relations $R \in \mathbf{Rel}_{\mathbb{R}^S_\omega}$ as

$$u(R) = \{(v_1, v_2) \text{ s.t. } v_1 = v_1' + v_1'',\ \ v_2 = v_2' + v_2'' \text{ and } (v_1', v_2') \in R,\ (v_1'', v_2'') \in R\}$$

$$\bullet(R) = \{(v_1, v_2) \text{ s.t. } v_1 = k \cdot v_1',\ \ v_2 = k \cdot v_2' \text{ and } (v_1', v_2') \in R\}.$$

Question 1.1 *Prove that c is sound with respect to b.*

Question 1.2 *Suppose to have an oracle to decide whether $(v_1, v_2) \in c(R)$ for all vectors $v_1, v_2 \in \mathbb{R}^S_\omega$ and relations $R \in \mathbf{Rel}_{\mathbb{R}^S_\omega}$. Consider the algorithm `HKC` for weighted automata seen at lesson. Prove that if `HKC`(v_1, v_2) returns true, then v_1 and v_2 accepts the same weighted language.*

2 Streams

Let $\mathbb{R}$ be the field of real numbers. Consider the following stream GSOS specification on $\mathbb{R}$.

$$\frac{}{n \xrightarrow{n} 0} \qquad \frac{x \xrightarrow{n} x' \qquad y \xrightarrow{m} y'}{x \oplus y \xrightarrow{n+m} x' \oplus y'} \qquad \frac{x \xrightarrow{n} x' \qquad y \xrightarrow{m} y'}{x \otimes y \xrightarrow{n \times m} (n \otimes y') \oplus (x' \otimes y)}$$

Question 2.1 *Use coinduction to prove associativity and commutativity of $\oplus$, that is for all streams $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^\omega$,*

$$(\sigma_1 \oplus \sigma_2) \oplus \sigma_3 = \sigma_1 \oplus (\sigma_2 \oplus \sigma_3) \quad \textit{and} \quad \sigma_1 \oplus \sigma_2 = \sigma_2 \oplus \sigma_1.$$

Question 2.2 *Use coinduction up-to and the previous result to prove distributivity of $\otimes$ over $\oplus$, that is for all streams $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^\omega$,*

$$\sigma_1 \otimes (\sigma_2 \oplus \sigma_3) = (\sigma_1 \otimes \sigma_2) \oplus (\sigma_1 \otimes \sigma_3).$$