From Equivalences to Metrics

Filippo Bonchi
PACE meeting
(BOLOGNA)
From Equivalences to Metrics

F. van Breugel, J. Worrell: A behavioural pseudometric for probabilistic transition systems. TCS 2005
F. van Breugel, James Worrell: Approximating and computing behavioural distances in probabilistic transition systems. TCS 2006
Motivations

$s$ and $s_\epsilon$ are NOT behaviorally equivalent...

...but for small $\epsilon$, they almost behave the same
From Equivalences to Distances

• Behavioural Equivalences are the foundations of qualitative reasoning
• Behavioural Distances are the foundations of quantitative reasoning
• A Behavioural Distance is a pseudo-Metric
  \[ d: S \times S \to [0, 1] \]
  that assigns to two systems the distance of their behaviours
• \( d(p, q) = 0 \) iff \( p \) is behaviourally equivalent to \( q \)
From Equivalences to Distances

- Behavioural Equivalences are the foundations of qualitative reasoning.
- Behavioural Distances are the foundations of quantitative reasoning.
- A Behavioural Distance is a pseudo-Metric $d:S \times S \rightarrow [0,1]$ that assigns to two systems the distance of their behaviours.

Does these systems behave the same?

Does a system satisfy a certain property?

How far apart is the behaviour of these systems?

How much closely does a system come to satisfy a certain property?
Plan of the Talk

1. Coalgebras in a nutshell
2. Probabilistic Systems
3. Behavioural (Pseudo-)Metrics
   – Coalgebras
   – Coinduction
   – Refinement Algorithm
   – Modal Logic
Coalgebras in a nutshell

At the blackboard
A functor $F$ induces:

1) Behavioural equivalence $\cong_F$

2) Coinduction Proof Principle:
$x \cong_F y$ iff $xRy$ for some $F$-bisimulation $R$

3) Partition $F$-Refinement Algorithm

4) An Hennessy-Milner Logic:
$x \cong_F y$ iff $f(x) = f(y)$ for all $F$-formulas $f$
**Functors**

*Set* is the category of sets and functions

\[ F::= \text{Id}, A, F+F, FxF, F^A, P(F), D(F) \]
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A Hierarchy of Probabilistic Systems Types

F. Bartels, A. Sokolova, E. de Vink: A hierarchy of probabilistic system types. TCS 2004
Markov Chains

\[ D(\text{Id}): \text{Set} \rightarrow \text{Set} \]
\[ D(S) = \{ \mu: S \rightarrow [0,1] | \mu[X] = 1, \text{spt}(\mu) \text{ finite} \} \]
\[ <S, \alpha: S \rightarrow D(S)> \]
\[ D(\text{Id}): \text{Set} \rightarrow \text{Set} \]
\[ D(S) = \{ \mu: S \rightarrow [0,1] \mid \mu[X] = 1, \text{spt} (\mu) \text{ finite} \} \]
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Markov Chains

\[ D(\text{Id}): \text{Set} \rightarrow \text{Set} \]
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\[ <S, \alpha: S \rightarrow D(S)> \]
Generative Systems

\[ D(A \times \text{Id}) : \text{Set} \rightarrow \text{Set} \]

\[ \langle S, \alpha : S \rightarrow D(A \times S) \rangle \]

\[ A = \{a, b\} \]
Reactive Systems

\[ D(Id)^A : \text{Set} \rightarrow \text{Set} \]

\[ <S, \alpha : S \rightarrow D(S)^A> \]

\[ A = \{a, b\} \]
\[ D(\text{Id}) + P(A \times \text{Id}): \text{Set} \rightarrow \text{Set} \]

\[ A = \{ a, b \} \]

\[ <S, \alpha : S \rightarrow D(S) + P(A \times S)> \]
Simple Probabilistic Automata (Simple Segala Systems)

\( P(A \times D(\text{Id})) : \text{Set} \rightarrow \text{Set} \) \( A = \{a, b\} \)

\(<S, \alpha : S \rightarrow P(A \times D(S))>\)
Probabilistic Automata
(Segala Systems)

$PD(A \times \text{id}): \text{Set} \rightarrow \text{Set}$

$A = \{a, b\}$

$<S, \alpha: S \rightarrow P D(A \times S)>$

![Diagram showing a probabilistic automaton with states $s_1$, $s_2$, $s_3$, $s_4$, $s_5$, and transitions labeled with probabilities.]
(Partial) Markov Chains

\[ D(1+\text{Id}) : \text{Set} \to \text{Set} \]

\[ <S, \pi : S \to D(1+S)> \]
(Partial) Markov Chains

\[ D(1 + \text{Id}) : \text{Set} \rightarrow \text{Set} \]

\[ <S, \pi : S \rightarrow D(1 + S)> \]

R is a Bisimulation iff whenever \( s_1 R s_2 \) then

For all equivalence classes E of R

\[ \sum_{s \in E} \pi(s_1, s) = \sum_{s \in E} \pi(s_2, s) \]
(Partial) Markov Chains
Partition Refinement
(Partial) Markov Chains
Partition Refinement
(Partial) Markov Chains
Partition Refinement

\[ s_1 \xrightarrow{1/3} s_2 \xrightarrow{2/3} s_3 \xrightarrow{1} t \xrightarrow{1} s_1 \]

\[ s_2 \xrightarrow{1} s_4 \xrightarrow{1/2} s_5 \xrightarrow{1/2} s_3 \]

\[ s_3 \xrightarrow{1} s_5 \xrightarrow{1} s_6 \]
(Partial) Markov Chains
Partition Refinement
(Partial) Markov Chains
Partition Refinement

\[ \begin{align*}
s_1 & \xrightarrow{1/3} s_2, \\
& \xrightarrow{1/2} s_4, \\
& \xrightarrow{1/2} s_6, \\
s_2 & \xrightarrow{2/3} s_3, \\
s_3 & \xrightarrow{1} s_4, \\
& \xrightarrow{1} s_5, \\
& \xrightarrow{1} s_6, \\
s_4 & \xrightarrow{1} t, \\
s_5 & \xrightarrow{1} t, \\
s_6 & \xrightarrow{1} t, \\
t & \xrightarrow{1} t 
\end{align*} \]
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   - Modal Logic
Functors

\textbf{Ms} is the category of metric spaces and non-expansive maps

\[ F := \text{Id}, \ A, \ F+F, \ FxF, \ F^A, \ P(F), \ D(F), \ \delta(F) \]

\( \delta \) is in \([0,1]\)
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Example of Coinduction

<table>
<thead>
<tr>
<th>d</th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2ε</td>
</tr>
<tr>
<td>s₂</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>t</td>
<td>2ε</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Post-fix point

\[ \Delta(d)(s,t) \leq \epsilon + 2\epsilon/3 \leq 2\epsilon = d(s,t) \]

\[ d_F(s,t) \leq 2\epsilon \]

Coupling of s and t

<table>
<thead>
<tr>
<th></th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
<th>\pi(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>s₂</td>
<td>0</td>
<td>1/3-\epsilon</td>
<td>\epsilon</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>s₃</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>\pi(t)</td>
<td>0</td>
<td>1/3-\epsilon</td>
<td>1/3+\epsilon</td>
<td>1/3</td>
<td></td>
</tr>
</tbody>
</table>
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Metric Refinement

<table>
<thead>
<tr>
<th>d0</th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>s₂</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>s₃</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Metric Refinement

\[
\begin{array}{c|cccc}
\text{d0} & s & s_2 & s_3 & t \\
\hline
s & 0 & 0 & 0 & 0 \\
s_2 & 0 & 0 & 0 & 0 \\
s_3 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{d1} & s & s_2 & s_3 & t \\
\hline
s & 0 & 0 & 1 & 0 \\
s_2 & 0 & 0 & 1 & 0 \\
s_3 & 1 & 1 & 1 & 1 \\
t & 0 & 0 & 1 & 0 \\
\end{array}
\]
## Metric Refinement

<table>
<thead>
<tr>
<th>d2</th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>1/3</td>
<td>1</td>
<td>ε</td>
</tr>
<tr>
<td>s₂</td>
<td>1/3</td>
<td>0</td>
<td>1</td>
<td>1/3+ε</td>
</tr>
<tr>
<td>s₃</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>t</td>
<td>ε</td>
<td>1/3+ε</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

![Diagram](https://via.placeholder.com/150x150.png?text=Diagram)
## Metric Refinement

### Table 1

<table>
<thead>
<tr>
<th></th>
<th>d2</th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>1/3</td>
<td>1</td>
<td>ε</td>
<td></td>
</tr>
<tr>
<td>s₂</td>
<td>1/3</td>
<td>0</td>
<td>1</td>
<td>1/3+ε</td>
<td></td>
</tr>
<tr>
<td>s₃</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>ε</td>
<td>1/3+ε</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>d3</th>
<th>s</th>
<th>s₂</th>
<th>s₃</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>1/3+1/9</td>
<td>1</td>
<td>ε+ε/3</td>
<td></td>
</tr>
<tr>
<td>s₂</td>
<td>1/3+1/9</td>
<td>0</td>
<td>1</td>
<td>1/3+ε+1/9+ε/3</td>
<td></td>
</tr>
<tr>
<td>s₃</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>ε+ε/3</td>
<td>1/3+ε+1/9+ε/3</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
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A Logical Characterization

• Modal Formulas $f$ are functions $f: S \rightarrow \{0, 1\}$

• Quantitative Formulas $f: S \rightarrow [0, 1]$

$$d(s_1, s_2) = \sup_f |f(s_1) - f(s_2)|$$

P. Panangaden et al.
Metrics for labeled Markov Processes, TCS 2004
A quantitative logic

\[ \varphi ::= \text{true} \mid \Diamond \varphi \mid \varphi \land \varphi \mid \neg \varphi \mid \varphi \oplus q \]

\[ \text{D}(1+1d): \text{Set} \to \text{Set} \]

\[ <S, \pi: S \to \text{D}(1+S)> \]

\[ \begin{align*}
[\text{true}]_\delta(s) &= 1 \\
[\Diamond \varphi]_\delta(s) &= \delta \sum_{s' \in S} \pi(s, s') [\varphi]_\delta(s') \\
[\varphi \land \psi]_\delta(s) &= \min\{[\varphi]_\delta(s), [\psi]_\delta(s)\} \\
[\neg \varphi]_\delta(s) &= 1 - [\varphi]_\delta(s) \\
[\varphi \oplus q]_\delta(s) &= \max\{[\varphi]_\delta(s) - q, 0\}
\end{align*} \]