From Equivalences to Metrics

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PACE meeting

(BOLOGNA)

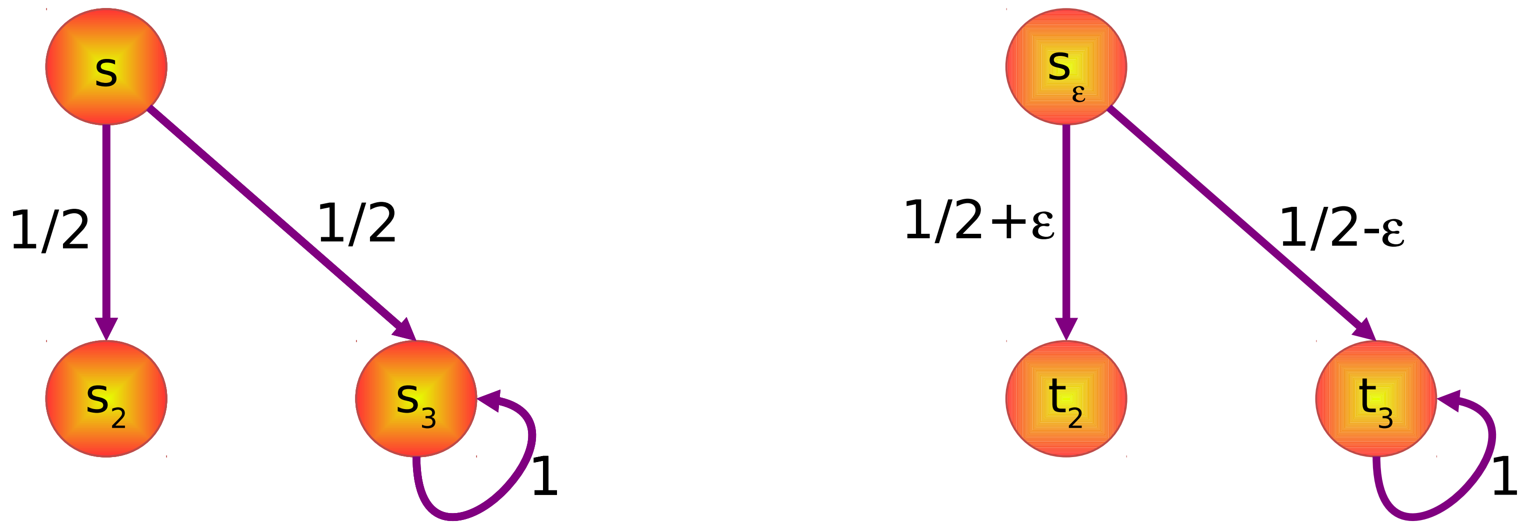
From Equivalences to Metrics

F. van Breugel, J. Worrell: A behavioural pseudometric for probabilistic transition systems. TCS 2005

F. van Breugel, James Worrell: Approximating and computing behavioural distances in probabilistic transition systems. TCS 2006

F. van Breugel, B. Sharma, J. Worrell: Approximating a Behavioural Pseudometric Without Discount for Probabilistic Systems. LMCS 2008

Motivations



s and s_ε are NOT behaviorally equivalent …

… but for small ε, they almost behave the same

From Equivalences to Distances

- Behavioural Equivalences are the foundations of qualitative reasoning
- Behavioural Distances are the foundations of quantitative resoning
- A Behavioural Distance is a pseudo-Metric

  $$d:SxS \rightarrow [0,1]$$

  that assigns to two systems the distance of their behaviours
- d(p,q)= 0 iff p is behaviourally equivalent to q

From Equivalences to Distances

- Behavioural Equivalences are the foundations of qualitative
- Behavio
  fou
- A
  p

d:SxS→[0,1

wo

b

is b

equiv nt to q

Does these systems behave the same?

Does a system satify a certain property?

How far apart is the behaviour of these systems?

How much closely does as system come to satisfy a certain property?

Plan of the Talk

1. Coalgebras in a nutshell
2. Probabilistic Systems
3. Behavioural (Pseudo-)Metrics
   - Coalgebras
   - Coinduction
   - Refinement Algorithm
   - Modal Logic

Coalgebras in a nutshell

At the blackboard

A functor F induces:

1) Behavioural equivalence $\cong_F$

2) Coinduction Proof Principle:

$x \cong_F y$ iff xRy for some F-bisimulation R

3) Partition F-Refinement Algorithm

4) An Hennessy-Milner Logic:

$x \cong_F y$ iff f(x)=f(y) for all F-formulas f

Functors

<u>**Set**</u> is the category of sets and functions

F::= **Id**, **A**, **F+F**, **FxF**, $\mathbf{F}^{\mathbf{A}}$, **P(F)**, **D(F)**

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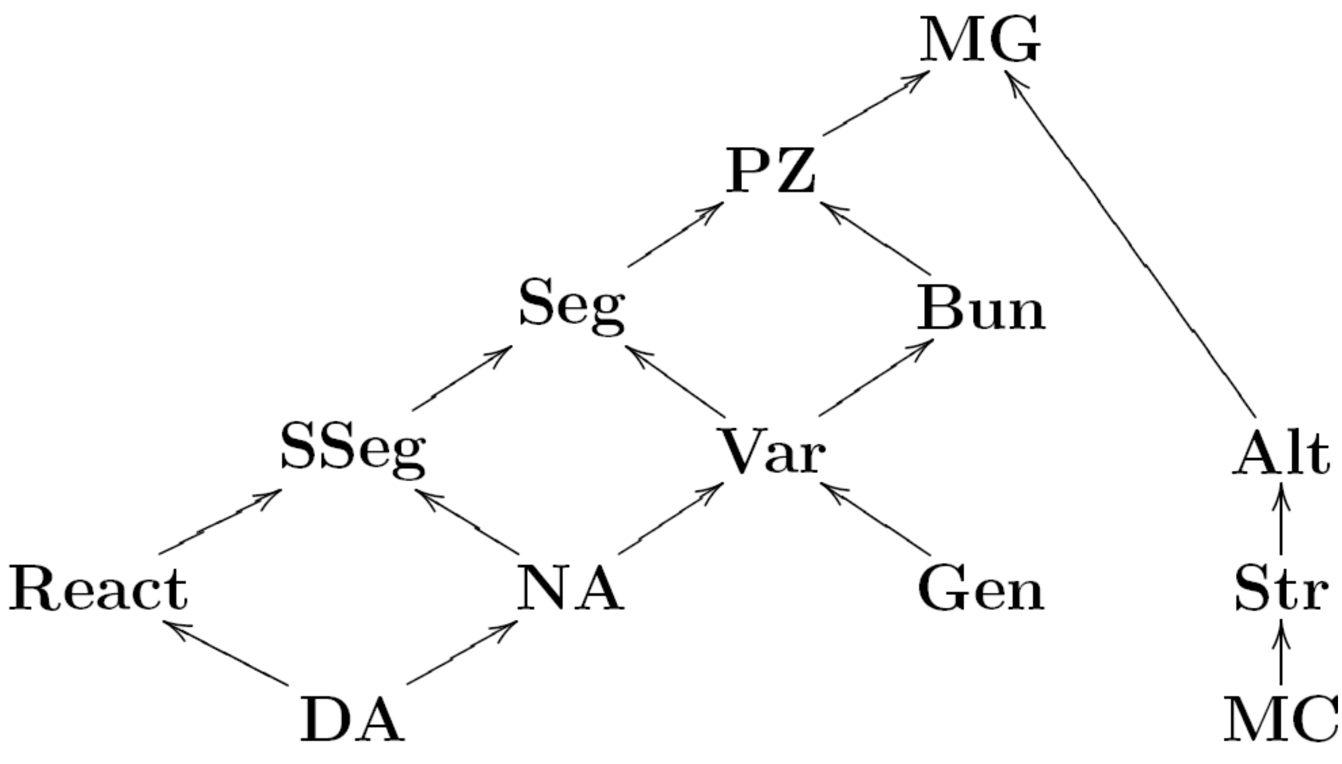
A Hierarchy of Probabilistic Systems Types



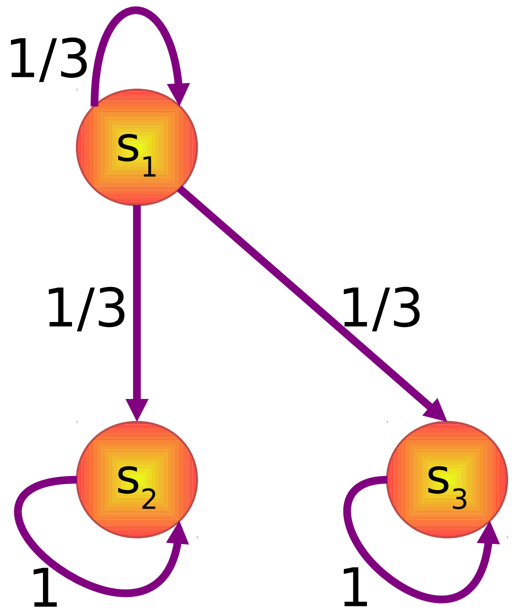
F. Bartels, A. Sokolova, E. de Vink: A hierarchy of probabilistic system types. TCS 2004

Markov Chains

D(Id):**Set**→**Set**

D(S)={μ:S→[0,1] | μ[X]=1, spt(μ) finite}

<S,α:S→**D**(S)>



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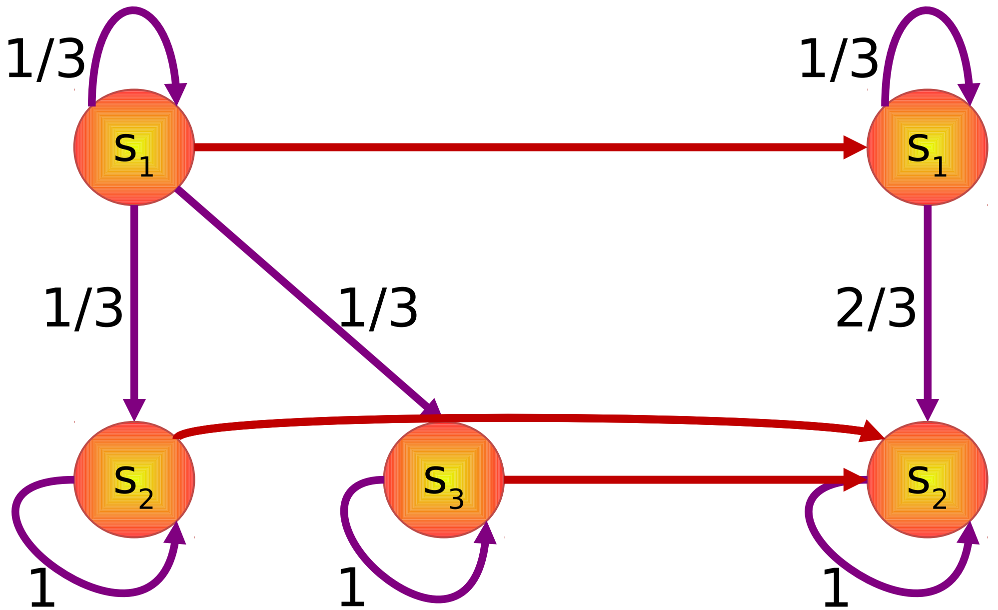
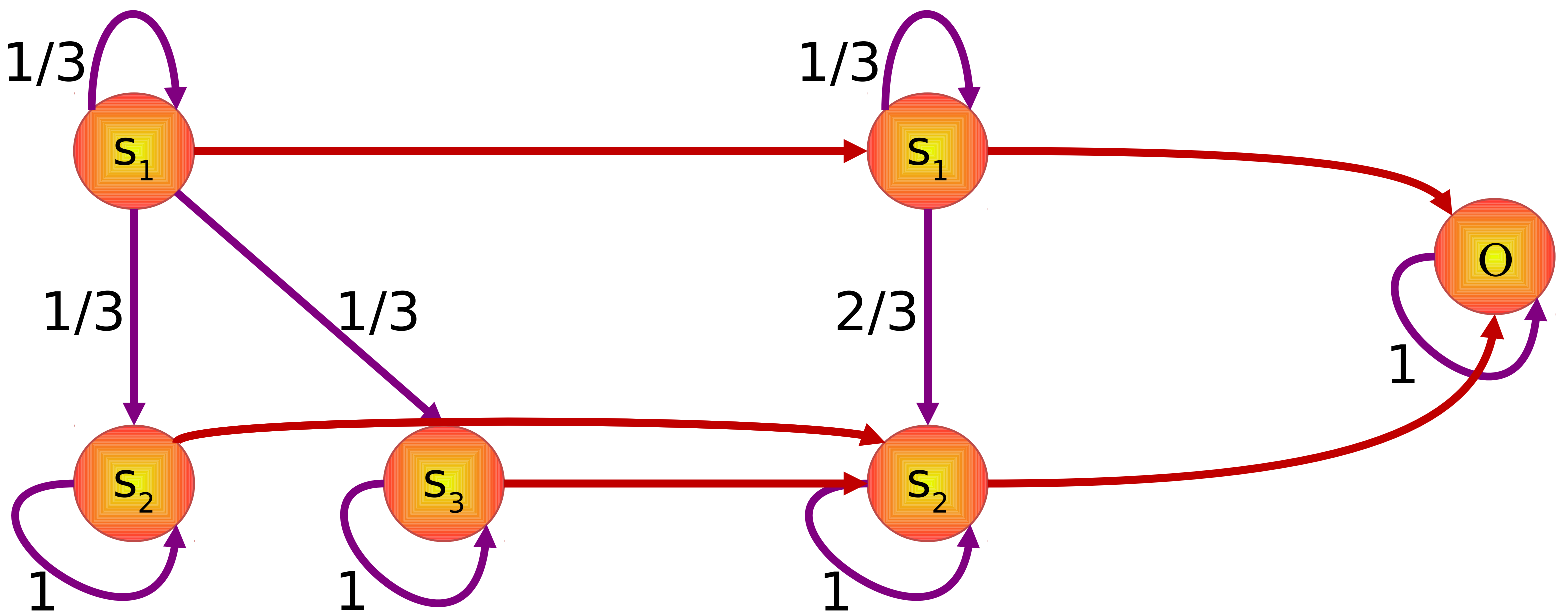


Markov Chains

D(Id): $\underline{\mathbf{Set}} \to \underline{\mathbf{Set}}$

$\mathbf{D}(S) = \{\mu : S \to [0,1] \mid \mu[X]=1, \ \mathrm{spt}(\mu) \text{ finite}\}$

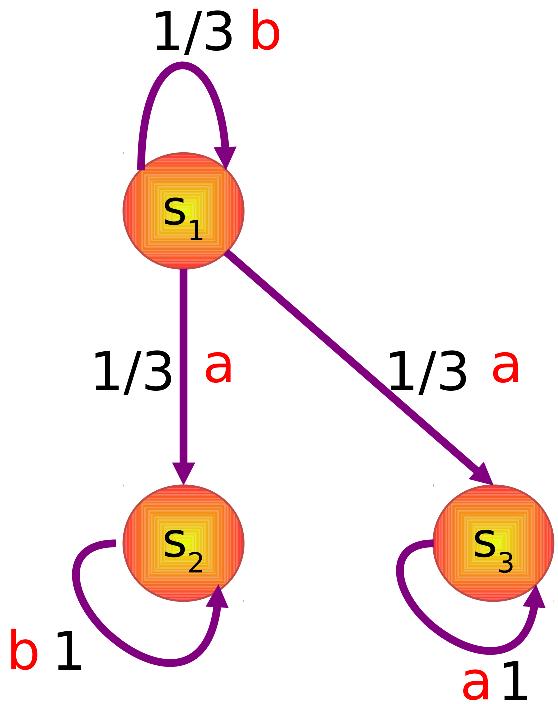
$\langle S, \alpha : S \to \mathbf{D}(S) \rangle$



Generative Systems

D(AxId):**Set**→**Set** A={a,b}

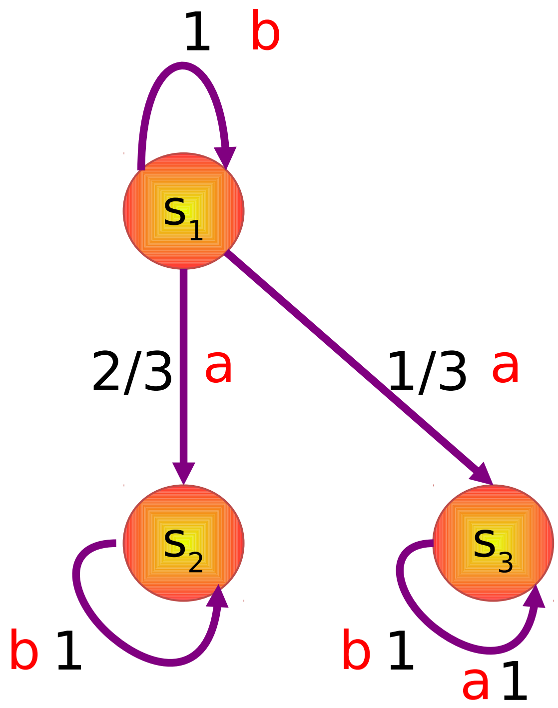
<S,α:S→**D(Ax**S**)**>



Reactive Systems

D(Id)^{A}:<u>Set</u>→<u>Set</u>   A={a,b}

<S,α:S→**D(**S**)**^{A}>

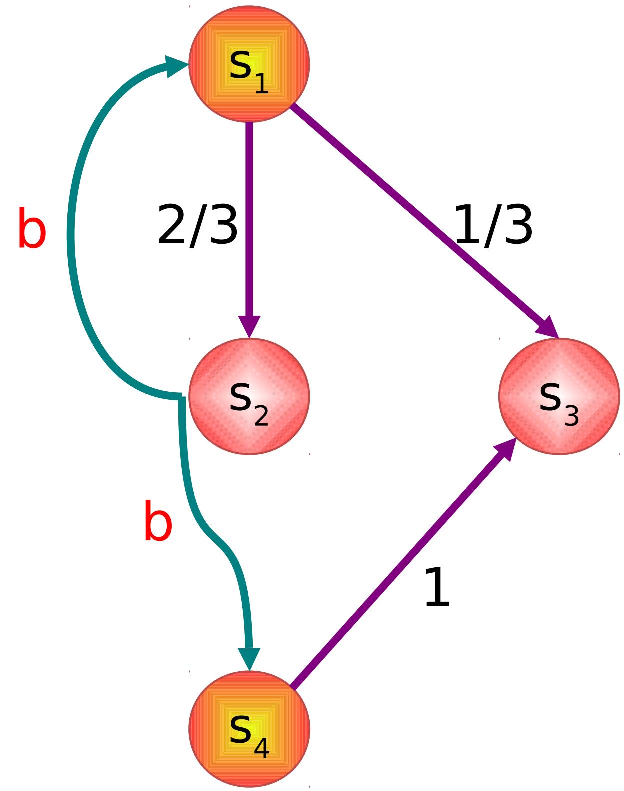


Alternating Systems

D(Id)+*P*(A×Id): <u>Set</u>→<u>Set</u>

A={a,b}

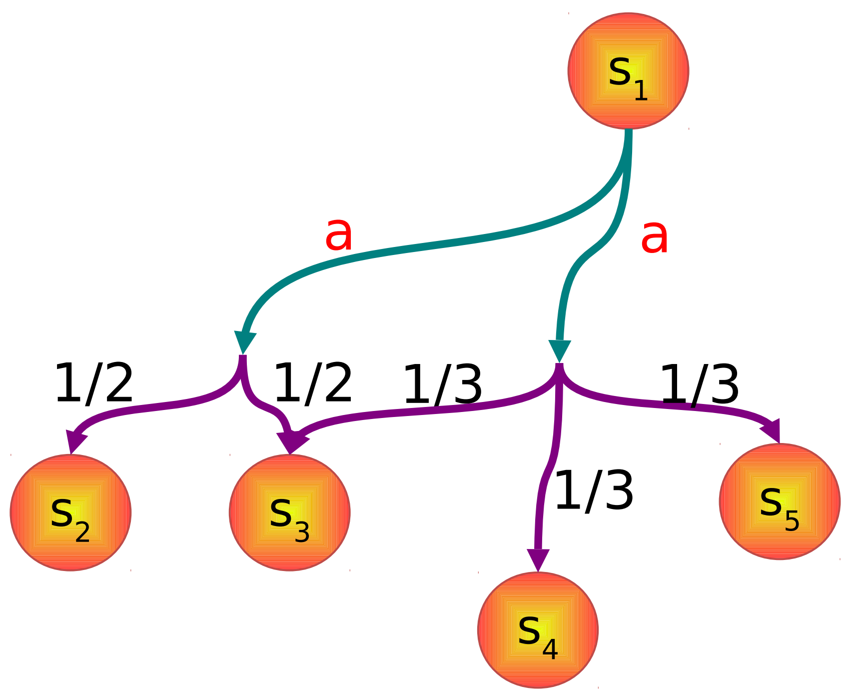
$\langle S, \alpha : S \to \mathbf{D}(S) + \boldsymbol{P}(\mathbf{A} \times S) \rangle$



Simple Probabilistic Automata (Simple Segala Systems)

***P*(A×D(Id)): <u>Set</u>→<u>Set</u>** A={a,b}

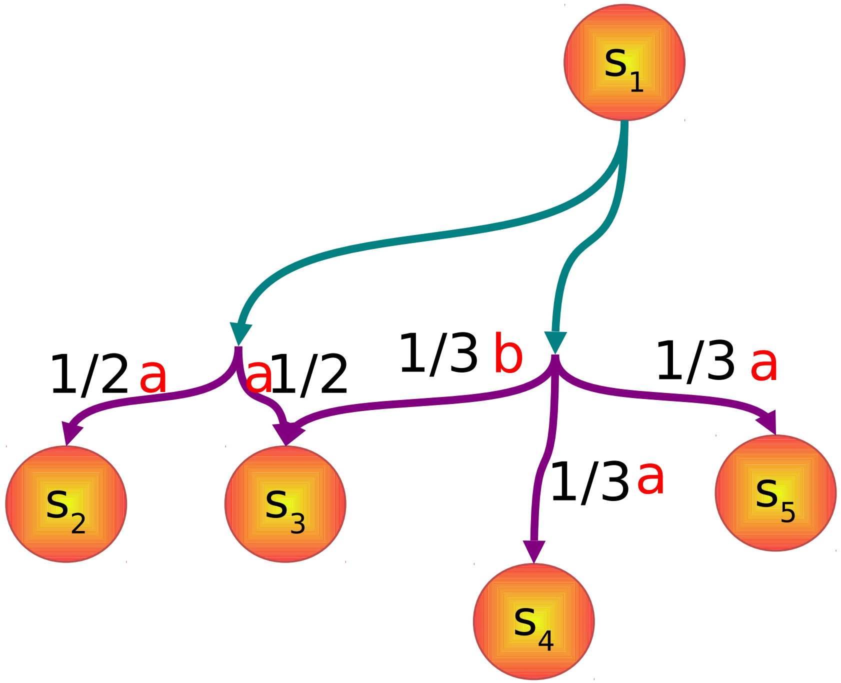
<S,α:S→***P* (A×D(**S**))**>



Probabilistic Automata
(Segala Systems)

***P*D(A×id): <u>Set</u>→<u>Set</u>** A={a,b}

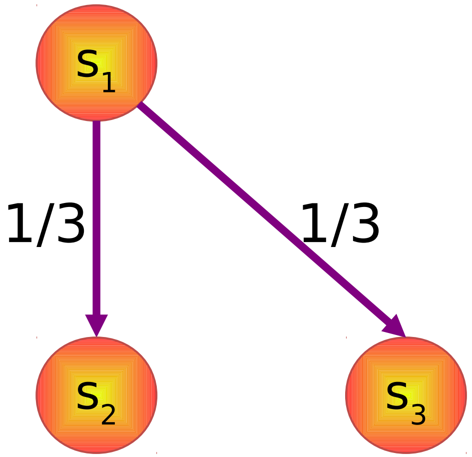
<S,α:S→***P* D(A×**S**)**>



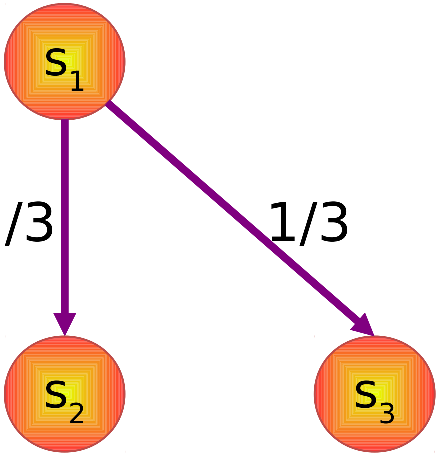
(Partial) Markov Chains

D(1+Id):**Set**→**Set**

<S,π:S→**D(1+**S**)**>



(Partial) Markov Chains Bisimulation



D(1+Id):**Set**→**Set**

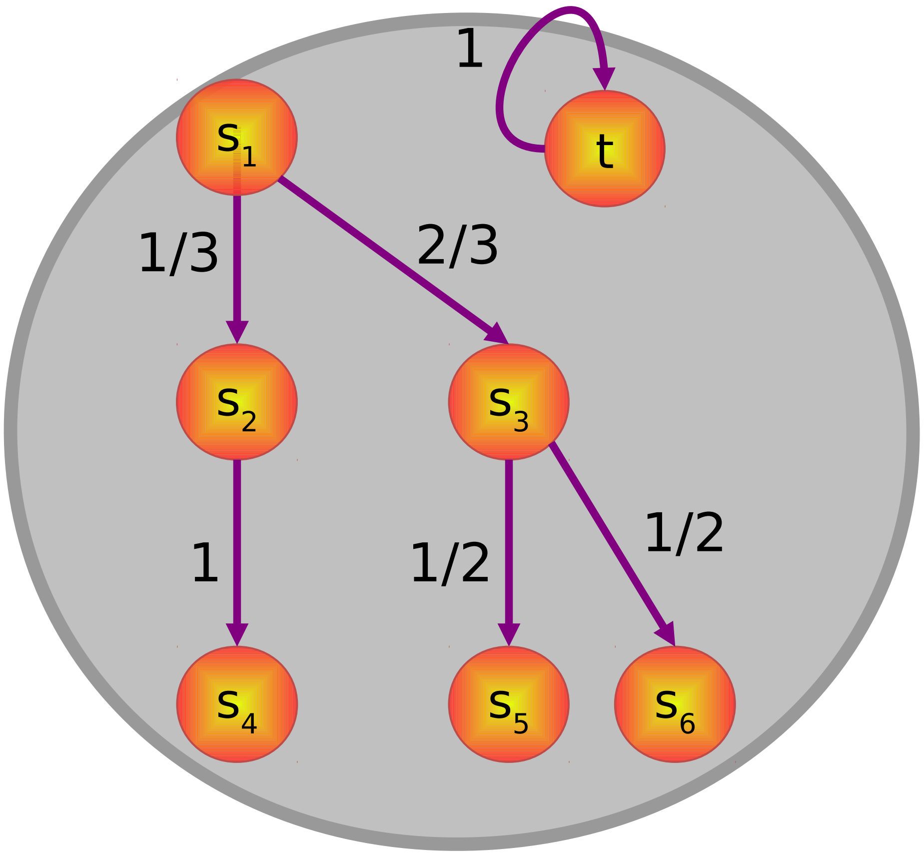
<S,π:S→**D(1+**S**)**>

R is a Bisimulation iff whenever $s_1 R s_2$ then

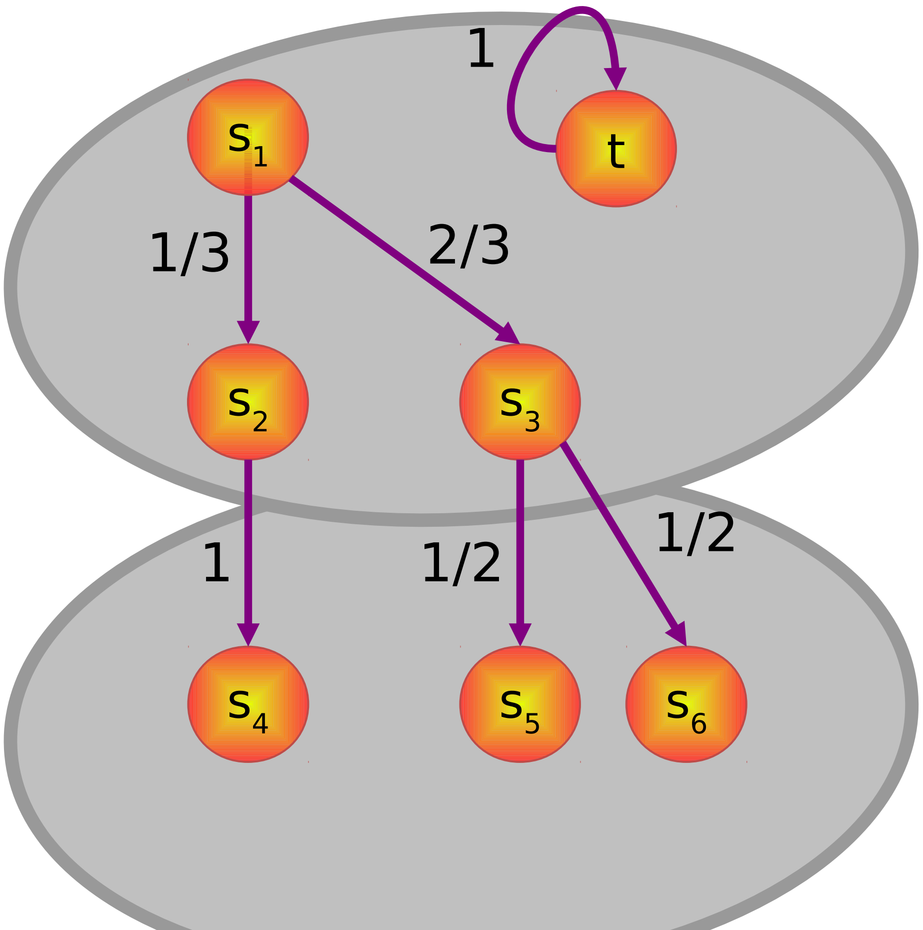
For all equivalence classes E of R

$$\sum_{s\in E} \pi(s_1, s) = \sum_{s\in E} \pi(s_2, s)$$

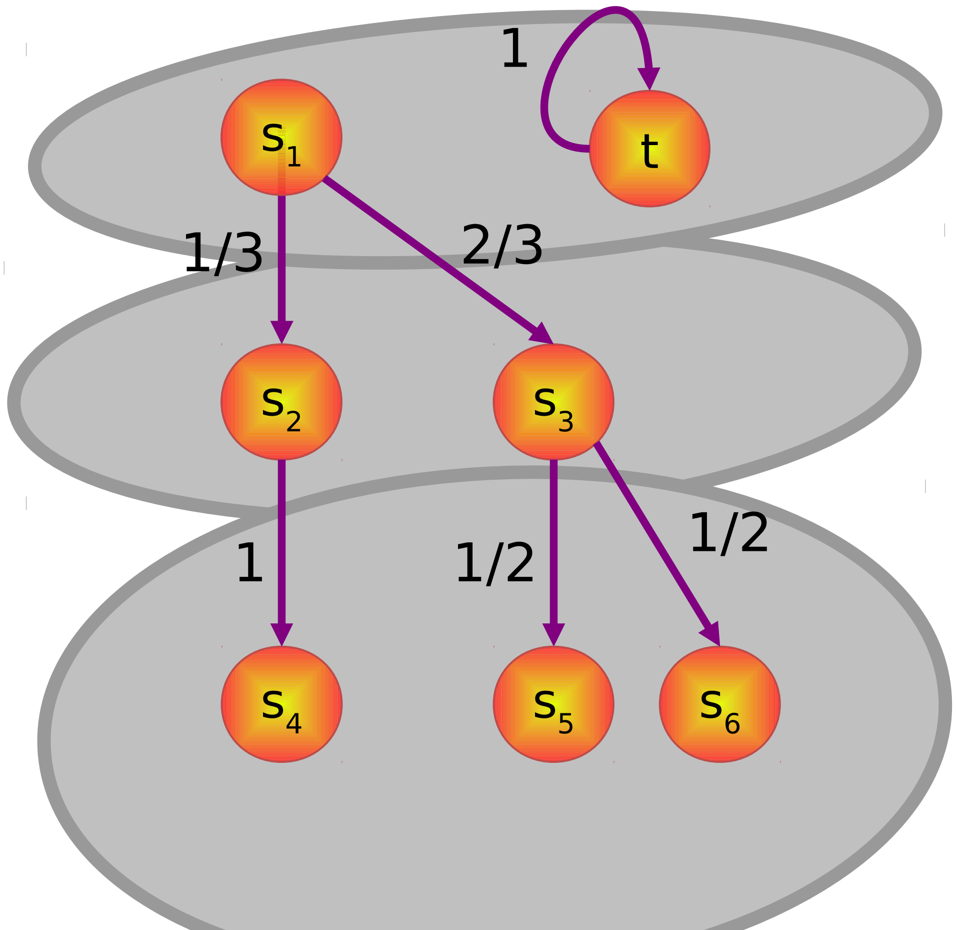
(Partial) Markov Chains Partition Refinement



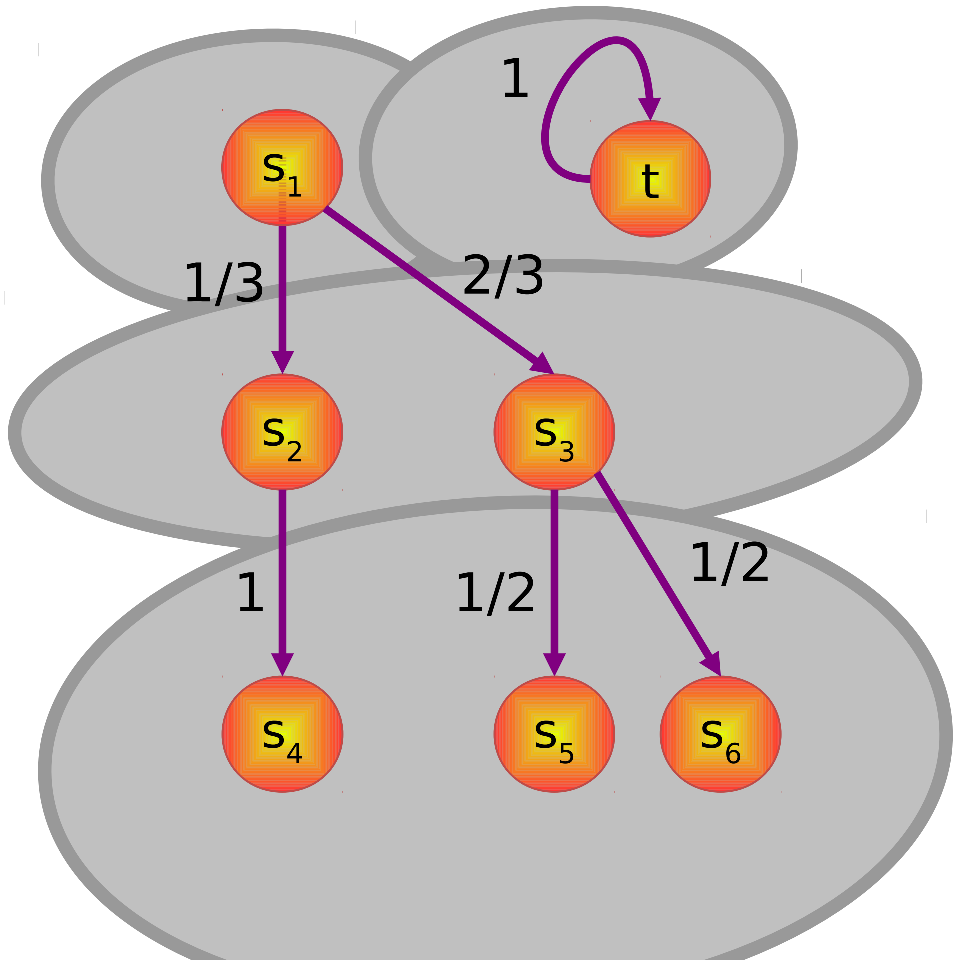
(Partial) Markov Chains
Partition Refinement



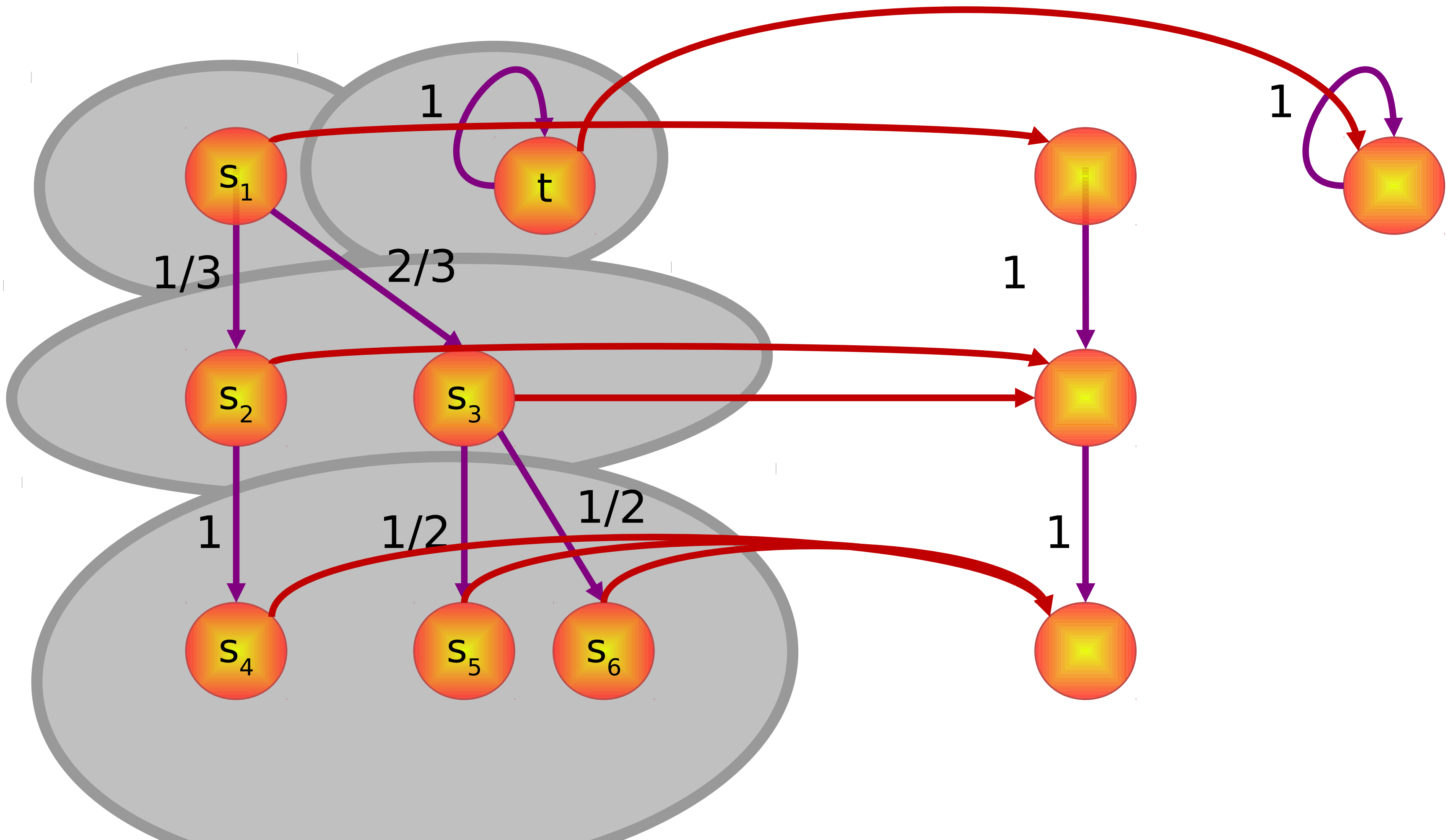
(Partial) Markov Chains
Partition Refinement



(Partial) Markov Chains
Partition Refinement



(Partial) Markov Chains Partition Refinement



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Functors

<u>Ms</u> is the category of metric spaces and non-expansive maps

F::= **Id**, **A**, **F+F**, **FxF**, $\mathbf{F}^{\mathbf{A}}$, **P(F)**, **D(F), δ(F)**

δ is in [0,1]

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Example of Coinduction

Post-fix point

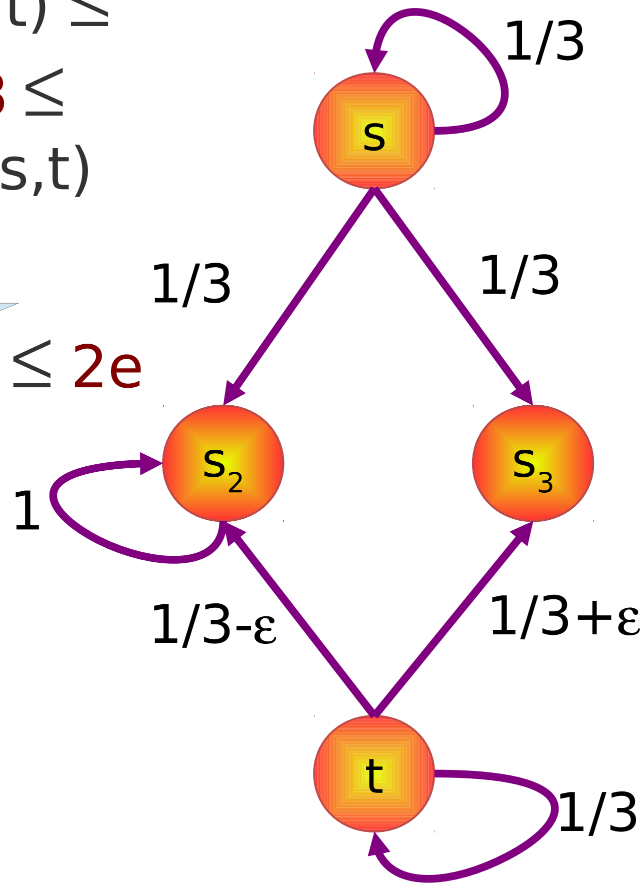
d	s	s_2	s_3	t
s	0	1	1	2ε
s_2	1	0	1	1
s_3	1	1	0	1
t	2ε	1	1	0

$\Delta(d)(s,t) \leq$

$\varepsilon + 2\varepsilon/3 \leq$

$2\varepsilon = d(s,t)$

$d_F(s,t) \leq 2e$



Coupling of s and t

	s	s_2	s_3	t	$\pi(s)$
s	0	0	0	1/3	1/3
s_2	0	$1/3-\varepsilon$	ε	0	1/3
s_3	0	0	1/3	0	1/3
t	0	0	0	0	0
$\pi(t)$	0	$1/3-\varepsilon$	$1/3+\varepsilon$	1/3	

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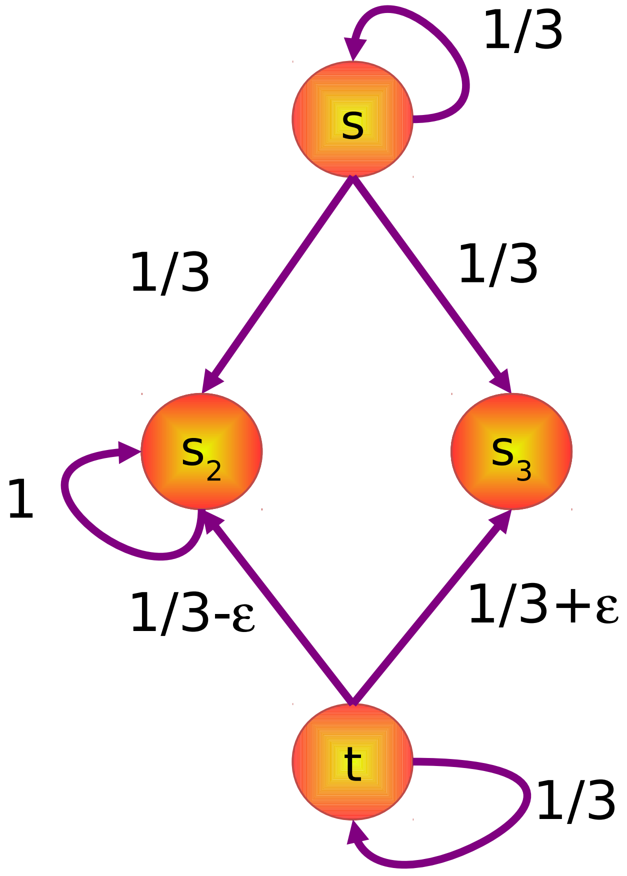
Metric Refinement

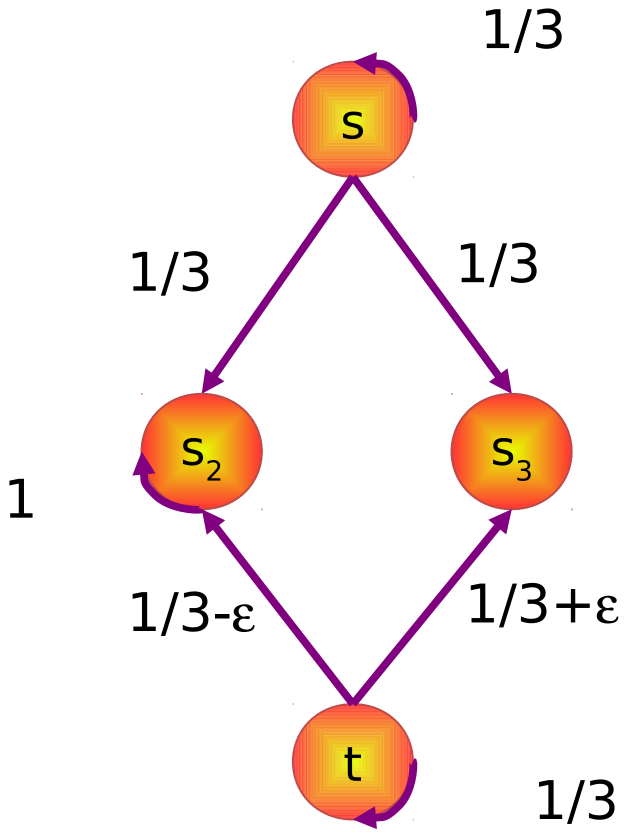
d0	s	s_2	s_3	t
s	0	0	0	0
s_2	0	0	0	0
s_3	0	0	0	0
t	0	0	0	0



Metric Refinement

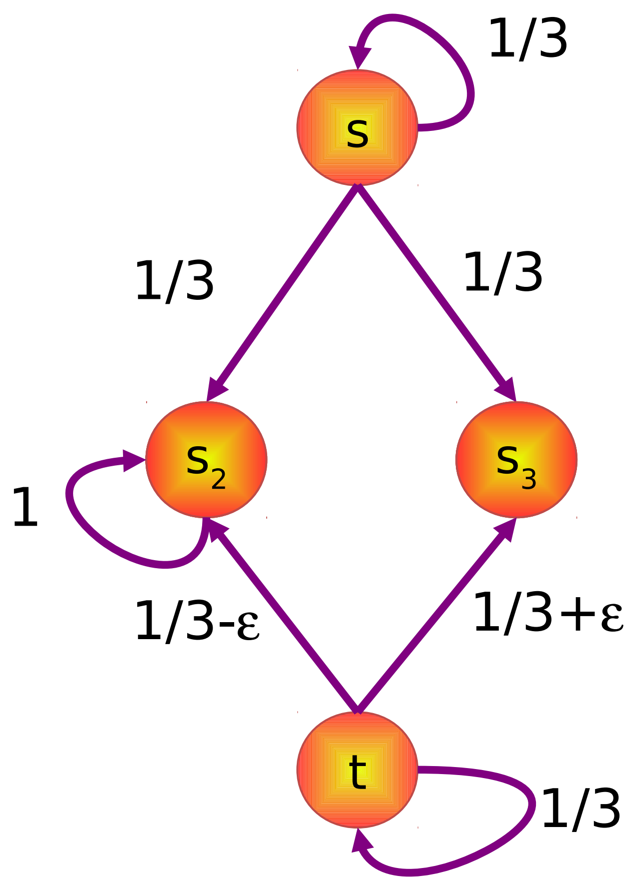
d0	s	s_2	s_3	t
s	0	0	0	0
s_2	0	0	0	0
s_3	0	0	0	0
t	0	0	0	0

d1	s	s_2	s_3	t
s	0	0	1	0
s_2	0	0	1	0
s_3	1	1	1	1
t	0	0	1	0



Metric Refinement

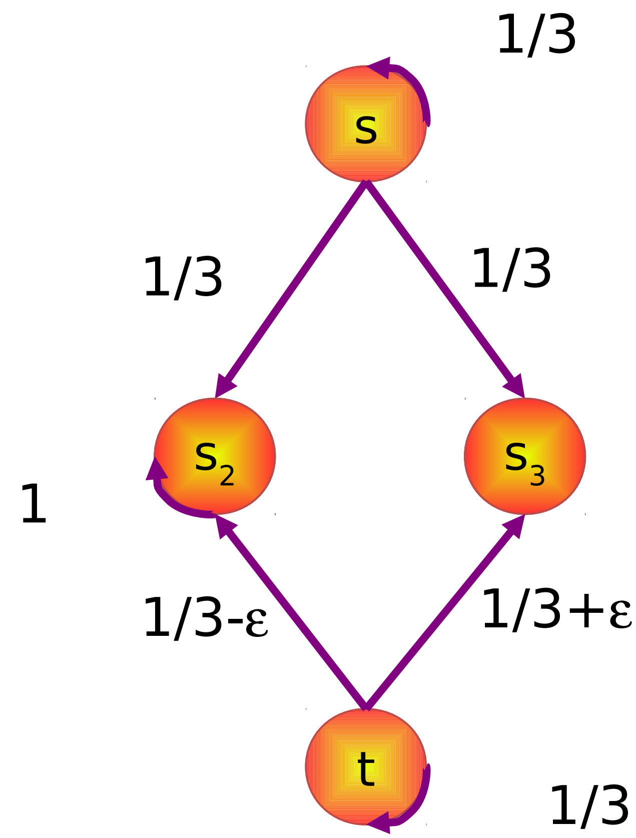
d2	s	s_2	s_3	t
s	0	1/3	1	ε
s_2	1/3	0	1	$1/3+\varepsilon$
s_3	1	1	0	1
t	ε	$1/3+\varepsilon$	1	0



Metric Refinement

d2	s	s_2	s_3	t
s	0	1/3	1	ε
s_2	1/3	0	1	1/3+ε
s_3	1	1	0	1
t	ε	1/3+ε	1	0

d3	s	s_2	s_3	t
s	0	1/3+1/9	1	ε+ε/3
s_2	1/3+1/9	0	1	1/3+ε+1/9+ε/3
s_3	1	1	0	1
t	ε+ε/3	1/3+ε+1/9+ε/3	1	0



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A Logical Characterization

- Modal Formulas f are functions $f:S \rightarrow \{0,1\}$
- Quantitative Formulas $f:S \rightarrow [0,1]$

$$d(s_1,s_2) = \sup_f |f(s_1)-f(s_2)|$$

P. Panangaden et al.
Metrics for labeled Markov Processes, TCS 2004

A quantitative logic

$$\varphi ::= \mathrm{true} \mid \Diamond\varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi \ominus q$$

D(1+Id):**<u>Set</u>→<u>Set</u>**

<S,π:S→**D(1+**S**)**>

$$
\begin{aligned}
[\![\mathrm{true}]\!]_\delta(s) &= 1\\
[\![\Diamond\varphi]\!]_\delta(s) &= \delta \textstyle\sum_{s' \in S} \pi(s,s')[\![\varphi]\!]_\delta(s')\\
[\![\varphi \wedge \psi]\!]_\delta(s) &= \min\{[\![\varphi]\!]_\delta(s), [\![\psi]\!]_\delta(s)\}\\
[\![\neg\varphi]\!]_\delta(s) &= 1 - [\![\varphi]\!]_\delta(s)\\
[\![\varphi \ominus q]\!]_\delta(s) &= \max\{[\![\varphi]\!]_\delta(s) - q, 0\}
\end{aligned}
$$