Probabilistic (Bi)simulation
(A Tutorial)

Yuxin Deng

Shanghai Jiao Tong University

http://basics.sjtu.edu.cn/∼yuxin/

Bologna, April 22, 2013

Outline

1. Probabilistic bisimulation
2. Justifying the lifting of relations
   (a) Justification by Kantorovich metric
   (b) Justification by network flow
3. *Metric characterisation of probabilistic bisimulation
4. *Algorithmic characterisation of probabilistic bisimulation
5. Logical characterisation of probabilistic bisimulation
6. Probabilistic simulations and testing preorders

* to be omitted

Probabilistic bisimulation

Probability distributions

- A (discrete) probability distribution over a countable set S is a function $\Delta : S \rightarrow [0,1]$ s.t. $\sum_{s \in S} \Delta(s) = 1$
- The support of Δ: $\lceil \Delta \rceil := \{s \in S | \Delta(s) > 0\}$
- $\mathcal{D}(S)$: the set of all distributions over S
- $\overline{s}$: the point distribution $\overline{s}(s) = 1$
- Given distributions $\Delta_1, ..., \Delta_n$, we form their linear combination $\sum_{i \in 1..n} p_i \cdot \Delta_i$, where $\forall i : p_i > 0$ and $\sum_{i \in 1..n} p_i = 1$.

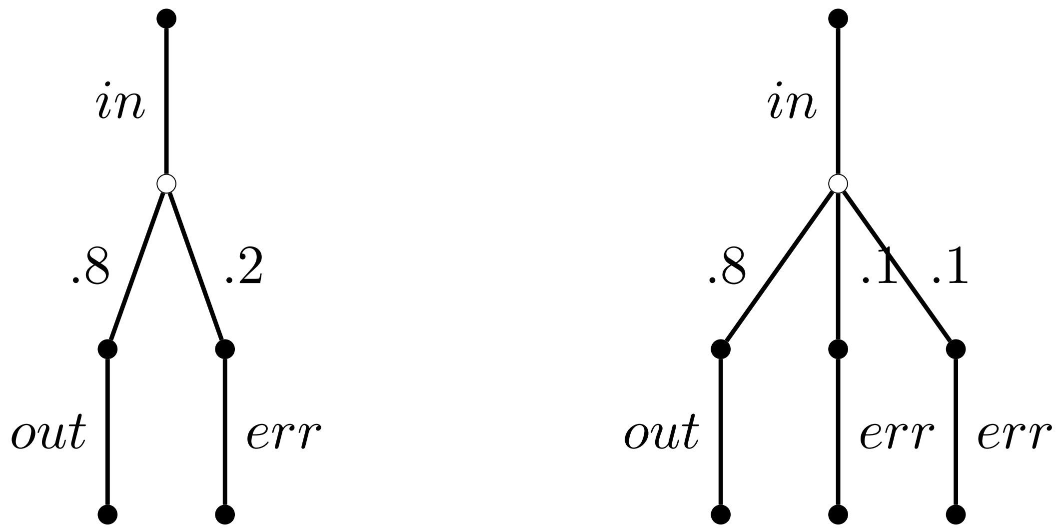
Probabilistic labelled transition systems

Def. A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, Act, \rightarrow\rangle$, where

1. S is a set of states
2. Act is a set of actions
3. $\rightarrow \;\subseteq\; S \times Act \times \mathcal{D}(S)$.

We usually write $s \xrightarrow{\alpha} \Delta$ in place of $(s, \alpha, \Delta) \in \rightarrow$. An LTS may be viewed as a degenerate pLTS that only uses point distributions.

Example



Lifting relations

Def. Let $\mathcal{R} \subseteq S \times T$ be a relation between sets S and T. Then $\mathcal{R}^\dagger \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is the smallest relation that satisfies:

1. $s \; \mathcal{R} \; t$ implies $\overline{s} \; \mathcal{R}^\dagger \; \overline{t}$
2. $\Delta_i \; \mathcal{R}^\dagger \; \Theta_i$ implies $(\sum_{i \in I} p_i \cdot \Delta_i) \; \mathcal{R}^\dagger \; (\sum_{i \in I} p_i \cdot \Theta_i)$ for any $p_i \in [0,1]$ with $\sum_{i \in I} p_i = 1$.

More discussion about the lifting operation later.

Bisimulation

Def. A binary relation $\mathcal{R} \subseteq S \times S$ is a simulation if whenever $s \mathrel{\mathcal{R}} t$:

- if $s \xrightarrow{a} \Delta$, there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta \mathrel{\mathcal{R}^\dagger} \Theta$.

The relation $\mathcal{R}$ is a bisimulation if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are simulations.
Bisimilarity, written $\sim$, is the union of all bisimulations.

Justifying the lifting of relations

Alternative ways of lifting (1/2)

Prop. $\Delta \mathcal{R}^\dagger \Theta$ if and only if

1. $\Delta = \sum_{i\in I} p_i \cdot \overline{s_i}$, where I is a countable index set and $\sum_{i\in I} p_i = 1$
2. For each $i \in I$ there is a state t_i such that $s_i \mathcal{R} t_i$
3. $\Theta = \sum_{i\in I} p_i \cdot \overline{t_i}$.

Alternative ways of lifting (2/2)

Prop.

1. Let Δ, Θ be distributions over S and $\mathcal{R}$ be an equivalence relation. Then
$$\Delta\ \mathcal{R}^\dagger\ \Theta \quad \text{iff} \quad \forall C \in S/\mathcal{R} : \Delta(C) = \Theta(C)$$
where $\Delta(C) = \sum_{s \in C} \Delta(s)$.

2. Let Δ and Θ be distributions over S and T, respectively. Then $\Delta\ \mathcal{R}^\dagger\ \Theta$ iff there exists a weight function $w : S \times T \to [0,1]$ such that
   (a) $\forall s \in S : \sum_{t \in T} w(s,t) = \Delta(s)$
   (b) $\forall t \in T : \sum_{s \in S} w(s,t) = \Theta(t)$
   (c) $\forall (s,t) \in S \times T : w(s,t) > 0 \Rightarrow s\ \mathcal{R}\ t$.

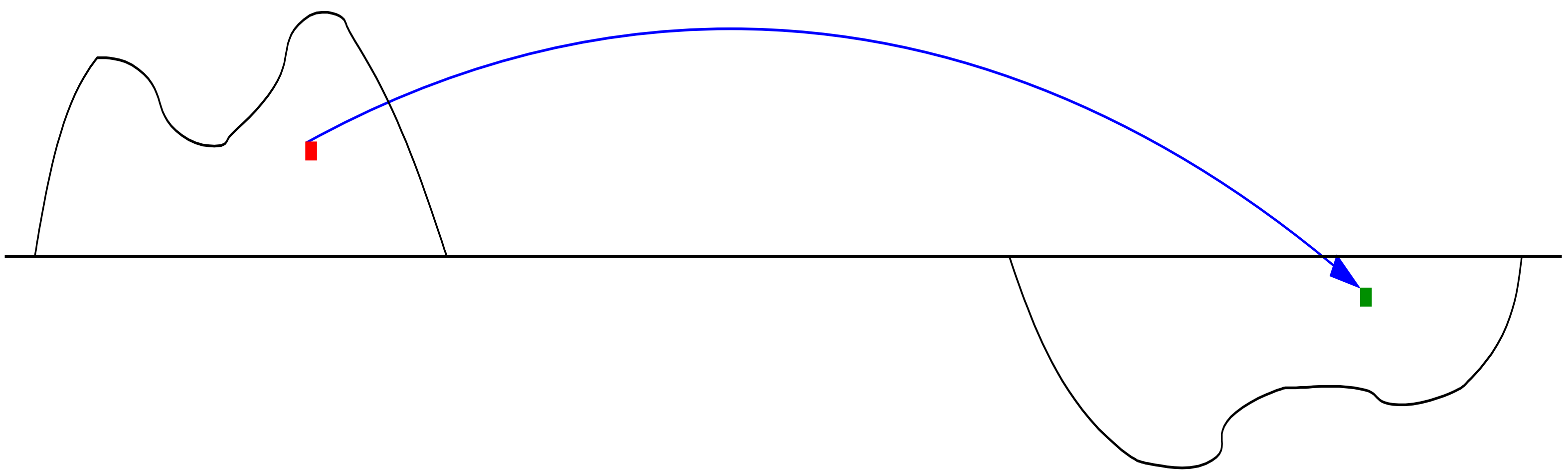
Relating the lifting operation with Kantorovich metric

The Kantorovich metric was motivated by the transportation problem.

The transportation problem

The original transportation problem (formulated by the French mathematician Gaspard Monge in 1781):

What's an optimal way of shovelling a pile of sand into a hole of the same volume?



Kantorovich metric

Def. Let (S, m) be a separable metric space. For any two Borel probability measures Δ and Θ on S, the *Kantorovich distance* between Δ and Θ is defined by

$$\hat{m}(\Delta, \Theta) = \sup\left\{\left|\int f d\Delta - \int f d\Theta\right| : ||f|| \leq 1\right\}.$$

where $||\cdot||$ is the *Lipschitz semi-norm* defined by $||f|| = \sup_{x \neq y} \frac{|f(x)-f(y)|}{m(x,y)}$ for a function $f : S \to \mathbb{R}$ with $\mathbb{R}$ being the set of all real numbers.

Kantorovich-Rubinstein Theorem

Write $M(\Delta, \Theta)$ for the set of all Borel probability measures on the product space $S \times S$ with marginal measures Δ and Θ, i.e. if $\Gamma \in M(\Delta, \Theta)$ then $\int_{y \in S} d\Gamma(x, y) = d\Delta(x)$ and $\int_{x \in S} d\Gamma(x, y) = d\Theta(y)$ hold.

Thm. If (S, m) is a separable metric space then for any two distributions $\Delta, \Theta \in \mathcal{D}(S)$ we have

$$\hat{m}(\Delta, \Theta) = \inf \left\{ \int m(x, y) d\Gamma(x, y) : \Gamma \in M(\Delta, \Theta) \right\}.$$

Interpretation of Kantorovich metric

Intuitively, a probability measure $\Gamma \in M(\Delta, \Theta)$ can be understood as a *transportation* from one unit mass distribution Δ to another unit mass distribution Θ. If the distance $m(x, y)$ represents the cost of moving one unit of mass from location x to location y then $\hat{m}(\Delta, \Theta)$ gives the optimal total cost of transporting the mass of Δ to Θ.

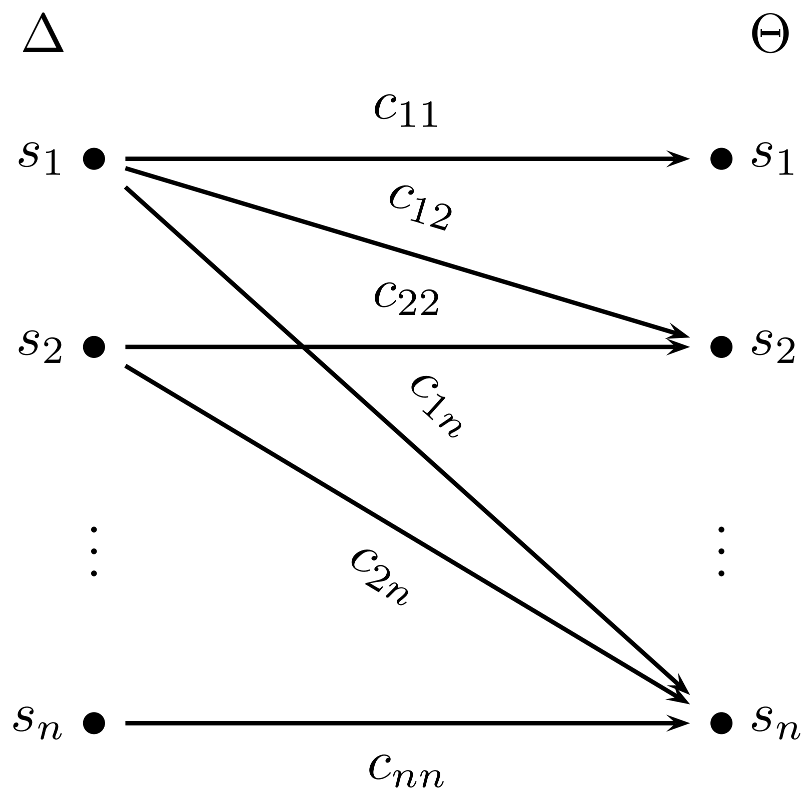
Discrete transportation problem

For two discrete distributions Δ and Θ with finite supports $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_l\}$, respectively, minimizing the total cost of a discretized version of the transportation problem reduces to the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n}\sum_{j=1}^{l}\Gamma(x_i,y_j)m(x_i,y_j) \\ \text{subject to} & \bullet\ \forall 1 \le i \le n : \sum_{j=1}^{l}\Gamma(x_i,y_j) = \Delta(x_i) \\ & \bullet\ \forall 1 \le j \le l : \sum_{i=1}^{n}\Gamma(x_i,y_j) = \Theta(y_j) \\ & \bullet\ \forall 1 \le i \le n, 1 \le j \le l : \Gamma(x_i,y_j) \ge 0. \end{array} \tag{1}$$

i.e. $\hat{m}(\Delta, \Theta)$ is the minimum value of problem (1).

Discrete transportation problem



c_{ij} stands for $m(s_i, s_j)$, for all i, j

Lifting relations vs. lifting metrics

Prop. Let R be a binary relation and m a pseudometric on a state space S satisfying

$$s \; R \; t \quad \text{iff} \quad m(s,t) = 0$$

for any $s, t \in S$. Then it holds that

$$\Delta \;\; R^{\dagger} \;\; \Theta \qquad \text{iff} \qquad \hat{m}(\Delta, \Theta) = 0$$

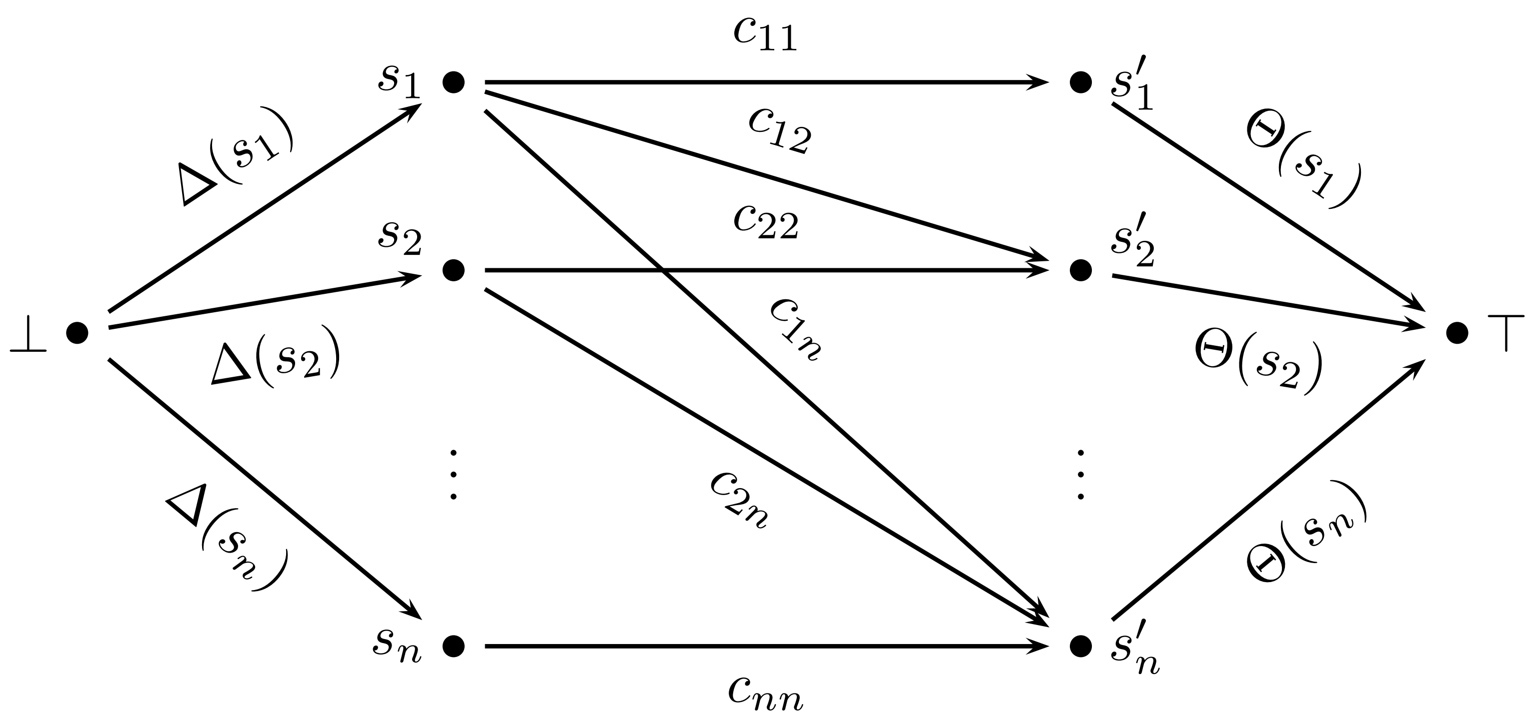
for any distributions $\Delta, \Theta \in \mathcal{D}(S)$.

Network

Def. A network is a tuple $\mathcal{N} = (N, E, \perp, \top, c)$ where

- (N, E) is a finite directed graph (i.e. N is a set of nodes and $E \subseteq N \times N$ is a set of edges)
- $\perp$ and $\top$ are the source and sink nodes respectively
- c is a capability function that assigns to each edge $(v, w) \in E$ a non-negative number $c(v, w)$.

Example



$$c_{ij} = 1 \text{ for all } i, j$$

Flow function

Def. A flow function f for $\mathcal{N}$ is a function that assigns to each edge e a real number $f(e)$ such that

- $0 \leq f(e) \leq c(e)$ for all edges e.
- For each node $v \in N \backslash \{\perp, \top\}$,

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

where $in(v)$ is the set of incoming edges to node v;
$out(v)$ the set of outgoing edges from node v.

Maximum flow

Def. The flow $F(f)$ of f is given by

$$F(f) \;=\; \sum_{e \in out(\perp)} f(e) - \sum_{e \in in(\perp)} f(e).$$

The maximum flow in $\mathcal{N}$ is the supremum (maximum) over the flows $F(f)$, where f is a flow function in $\mathcal{N}$.

The network $\mathcal{N}(\Delta, \Theta, \mathcal{R})$

Def. Let $S' = \{s' \mid s \in S\}$ and $\bot, \top$ are two new states with $\bot, \top \notin S \cup S'$. For any $\Delta, \Theta \in \mathcal{D}(S)$ and $\mathcal{R} \subseteq S \times S$, we construct the following network $\mathcal{N}(\Delta, \Theta, \mathcal{R}) = (N, E, \bot, \top, c)$.

- $N = S \cup S' \cup \{\bot, \top\}$.
- $E = \{(s, t') \mid (s, t) \in \mathcal{R}\} \cup \{(\bot, s) \mid s \in S\} \cup \{(s', \top) \mid s \in S\}$.
- c is defined by $c(\bot, s) = \Delta(s)$, $c(t', \top) = \Theta(t)$ and $c(s, t') = 1$ for all $s, t \in S$.

Relating the Lifting operation with network flow

Lem. [Baier et al., 2000] The following statements are equivalent.

1. There exists a weight function w for (Δ, Θ) with respect to $\mathcal{R}$.
2. The maximum flow in $\mathcal{N}(\Delta, \Theta, \mathcal{R})$ is 1.

Cor. $\Delta\ \mathcal{R}^\dagger\ \Theta$ iff the maximum flow in $\mathcal{N}(\Delta, \Theta, \mathcal{R})$ is 1.

Metric characterisation of bisimulation

Algorithmic characterisation of bisimulation

Logical characterisation of bisimulation

Adequacy and expressivity

Let $\mathcal{L}$ be a logic. The set of formulae that state s satisfies is denoted by $\mathcal{L}(s)$. Then $s =^{\mathcal{L}} t$ iff $\mathcal{L}(s) = \mathcal{L}(t)$.

- The logic $\mathcal{L}$ is adequate w.r.t. $\sim$ on a pLTS if for any states s and t,

$$s =^{\mathcal{L}} t \ \text{ iff } \ s \sim t.$$

- The logic $\mathcal{L}$ is expressive w.r.t. $\sim$ on a pLTS if for each state s there exists a characteristic formula $\varphi_s \in \mathcal{L}$ such that, for any states s and t,

$$t \models \varphi_s \ \text{ iff } \ s \sim t.$$

An adequate logic

$$
\begin{array}{rcl}
\varphi & := & \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \psi \mid \neg \varphi \\
\psi & := & \bigoplus_{i \in I} p_i \cdot \varphi_i
\end{array}
$$

- $s \models \top$ for all $s \in S$.
- $s \models \varphi_1 \wedge \varphi_2$ if $s \models \varphi_i$ for $i = 1, 2$.
- $s \models \langle a \rangle \psi$ if for some $\Delta \in \mathcal{D}(S)$, $s \xrightarrow{a} \Delta$ and $\Delta \models \psi$.
- $s \models \neg \varphi$ if it is not the case that $s \models \varphi$.
- $\Delta \models \bigoplus_{i \in I} p_i \cdot \varphi_i$ if there are $\Delta_i \in \mathcal{D}(S)$, for all $i \in I, t \in \lceil \Delta_i \rceil$, with $t \models \varphi_i$, such that $\Delta = \sum_{i \in I} p_i \cdot \Delta_i$.

Thm. $s \sim t$ iff $s =^{\mathcal{L}} t$.

Probabilistic modal μ-calculus (1/2)

Let Var be a set of variables. We define a set $\mathcal{L}_\mu$ of modal formulae in positive normal form:

$$\begin{array}{rcl} \varphi & := & \langle a\rangle\varphi \mid [a]\varphi \mid \bigwedge_{i\in I}\varphi_i \mid \bigvee_{i\in I}\varphi_i \mid X \mid \mu X.\varphi \mid \nu X.\varphi \\ \psi & := & \bigoplus_{i\in I} p_i\cdot\varphi_i \end{array}$$

where $a \in Act$, I is an finite index set and $\sum_{i\in I} p_i = 1$. Let $\bigwedge_{i\in\emptyset}\varphi_i = \top$ and $\bigvee_{i\in\emptyset}\varphi_i = \bot$.

Probabilistic modal μ-calculus (2/2)

Let $Env = \{ \rho \mid \rho : Var \to \mathcal{P}(S) \}$

$[\![\ ]\!] : \mathcal{L}_\mu \to Env \to \mathcal{P}(S)$

$$
\begin{array}{rcl}
[\![\top]\!]_\rho & = & S \\
[\![\bot]\!]_\rho & = & \emptyset \\
[\![\bigwedge_{i\in I} \varphi_i]\!]_\rho & = & \bigcap_{i\in I} [\![\varphi_i]\!]_\rho \\
[\![\bigvee_{i\in I} \varphi_i]\!]_\rho & = & \bigcup_{i\in I} [\![\varphi_i]\!]_\rho \\
[\![\langle a\rangle\psi]\!]_\rho & = & \{ s \in S \mid \exists\Delta : s \xrightarrow{a} \Delta \ \wedge\ \Delta \in [\![\psi]\!]_\rho \} \\
[\![[a]\varphi]\!]_\rho & = & \{ s \in S \mid \forall\Delta : s \xrightarrow{a} \Delta \ \Rightarrow\ \Delta \in [\![\psi]\!]_\rho \} \\
[\![X]\!]_\rho & = & \rho(X) \\
[\![\mu X.\varphi]\!]_\rho & = & \bigcap\{ V \subseteq S \mid [\![\varphi]\!]_{\rho[X\mapsto V]} \subseteq V \} \\
[\![\nu X.\varphi]\!]_\rho & = & \bigcup\{ V \subseteq S \mid [\![\varphi]\!]_{\rho[X\mapsto V]} \supseteq V \} \\
[\![\bigoplus_{i\in I} p_i \cdot \varphi_i]\!]_\rho & = & \{ \Delta \in \mathcal{D}(S) \mid \Delta = \bigoplus_{i\in I} p_i \cdot \Delta_i \ \wedge\ \forall i \in I, \forall t \in \lceil\Delta_i\rceil : t \in [\![\varphi_i]\!]_\rho \}
\end{array}
$$

Equation system of formulae

Let E be a closed equation systems of formulae.

$$\begin{aligned} E : X_1 &= \varphi_1 \\ &\vdots \\ X_n &= \varphi_n \end{aligned}$$

E viewed as a function $E : \mathit{Var} \to \mathcal{L}_\mu$ defined by $E(X_i) = \varphi_i$ for $i = 1, ..., n$ and $E(Y) = Y$ for other variables $Y \in \mathit{Var}$.

Def. An environment ρ is a solution of E if $\forall i : [\![X_i]\!]_\rho = [\![\varphi_i]\!]_\rho$.

Existence of solutions

1. The set Env with the partial order $\leq$ given by

$$\rho \leq \rho' \text{ iff } \forall X \in Var : \rho(X) \subseteq \rho'(X)$$

forms a complete lattice.

2. The equation functional $\mathcal{E} : Env \to Env$ given by

$$\mathcal{E} := \lambda\rho.\lambda X.[\![E(X)]\!]_\rho$$

is monotonic.

3. The Knaster-Tarski fixpoint theorem guarantees existence of solutions, and the largest solution

$$\rho_E := \bigsqcup \{\, \rho \mid \rho \leq \mathcal{E}(\rho) \,\}$$

Characteristic equation system

Def. Given a finite state pLTS, its characteristic equation system consists of one equation for each state $s_1, ..., s_n \in S$.

$$\begin{array}{rcl} E : X_{s_1} & = & \varphi_{s_1} \\ & \vdots & \\ X_{s_n} & = & \varphi_{s_n} \end{array}$$

where

$$\varphi_s := \Big(\bigwedge_{s \xrightarrow{a} \Delta} \langle a \rangle X_\Delta\Big) \wedge \Big(\bigwedge_{a \in Act} [a] \bigvee_{s \xrightarrow{a} \Delta} X_\Delta\Big)$$

with $X_\Delta := \bigoplus_{s \in \lceil \Delta \rceil} \Delta(s) \cdot X_s$.

Thm. If E is a characteristic equation system then $s \sim t$ iff $t \in \rho_E(X_s)$.

Characteristic formulae

- Rule 1: $E \rightarrow F$
- Rule 2: $E \rightarrow G$
- Rule 3: $E \rightarrow H$ if $X_n \notin fv(\varphi_1, ..., \varphi_n)$

$$
\begin{array}{rcl}
E : X_1 & = & \varphi_1 \\
 & \vdots & \\
X_{n-1} & = & \varphi_{n-1} \\
X_n & = & \varphi_n
\end{array}
\qquad
\begin{array}{rcl}
F : X_1 & = & \varphi_1 \\
 & \vdots & \\
X_{n-1} & = & \varphi_{n-1} \\
X_n & = & \nu X_n . \varphi_n
\end{array}
\qquad
\begin{array}{rcl}
G : X_1 & = & \varphi_1[\varphi_n / X_n] \\
 & \vdots & \\
X_{n-1} & = & \varphi_{n-1}[\varphi_n / X_n] \\
X_n & = & \varphi_n
\end{array}
\qquad
\begin{array}{c}
H : X_1 = \\
\vdots \\
X_{n-1} = \varphi_n
\end{array}
$$

Figure 1: Transformation rules

Thm. Given a characteristic equation system E, there is a characteristic formula φ_s such that $\rho_E(X_s) = [\![\varphi_s]\!]$ for any state s.

Probabilistic simulations

Simulations

simu. $(\lhd_S)$

$$
\begin{array}{ccc}
s & \xrightarrow{a} & \Delta \\
\mathcal{R} & & \mathcal{R}^{\dagger} \\
\Theta & \stackrel{\hat{a}}{\Longrightarrow} & \Theta'
\end{array}
$$

failure simu. $(\lhd_{FS})$

$$
\begin{array}{cccc}
s & (\stackrel{A}{\not\longrightarrow}) & \xrightarrow{a} & \Delta \\
\mathcal{R} & & & \mathcal{R}^{\dagger} \\
\Theta & \Longrightarrow \Theta' \stackrel{A}{\not\longrightarrow} & \stackrel{\hat{a}}{\Longrightarrow} & \Theta'
\end{array}
$$

Overview of results for finitary processes

$$
\begin{array}{ccccccccccccc}
 & & & & \sqsubseteq^n_{\mathbf{nr}\mathrm{may}} & \overset{[1]}{=} & \sqsubseteq^1_{\mathtt{may}} & \overset{[1]}{=} & \sqsubseteq^n_{\mathtt{may}} & \overset{[2]}{=} & \sqsubseteq_S \\
 & & & & \vdots & & & & & & \\
 & & & & \vdots & & & & & & \\
(\sqsubseteq^n_{\mathbf{rr}\mathrm{may}})^{-1} & \overset{[3]}{=} & \sqsubseteq^n_{\mathbf{rr}\mathrm{must}} & \overset{[3]}{=} & \sqsubseteq^n_{\mathbf{nr}\mathrm{must}} & \overset{[1]}{=} & \sqsubseteq^1_{\mathtt{must}} & \overset{[1]}{=} & \sqsubseteq^n_{\mathtt{must}} & \overset{[2]}{=} & \sqsubseteq_{FS}
\end{array}
$$

The symbol $=$ between two relations means that they coincide, while a vertical dotted line between two relations denotes that the relation below is finer than the relation above if divergence is absent.

[1]: [ESOP'07]; [2]: [LICS'07, CONCUR'09]; [3]: [QAPL'11] (for convergent processes)

A general testing scenario

Assume

- a set of processes $\mathcal{P}roc$,
- a set of tests $\mathcal{T}$,
- a set of outcomes $\mathcal{O}$, results of applying a test to a process
- a function $\mathcal{A} : \mathcal{T} \times \mathcal{P}roc \to \mathcal{P}^{+}_{fin}(\mathcal{O})$, to apply a test to a process
- $\mathcal{O}$ is endowed with a partial order, with $o_1 \leq o_2$ meaning o_2 is a better outcome than o_1.

Testing preorders

Comparing subsets of $\mathcal{O}$ with the Hoare or Smyth preorders.

Def. For $O_1, O_2 \in \mathcal{P}^+_{\mathit{fin}}(\mathcal{O})$

$$O_1 \leq_{\mathrm{Ho}} O_2 \quad \text{if} \quad \forall o_1 \in O_1\ \exists o_2 \in O_2 : o_1 \leq o_2$$

$$O_1 \leq_{\mathrm{Sm}} O_2 \quad \text{if} \quad \forall o_2 \in O_2\ \exists o_1 \in O_1 : o_1 \leq o_2.$$

For $P, Q \in \mathcal{P}roc$

$$P \sqsubseteq_{\mathrm{may}} Q \quad \text{if} \quad \mathcal{A}(T, P) \leq_{\mathrm{Ho}} \mathcal{A}(T, Q) \quad \text{for every test } T$$

$$P \sqsubseteq_{\mathrm{must}} Q \quad \text{if} \quad \mathcal{A}(T, P) \leq_{\mathrm{Sm}} \mathcal{A}(T, Q) \quad \text{for every test } T.$$

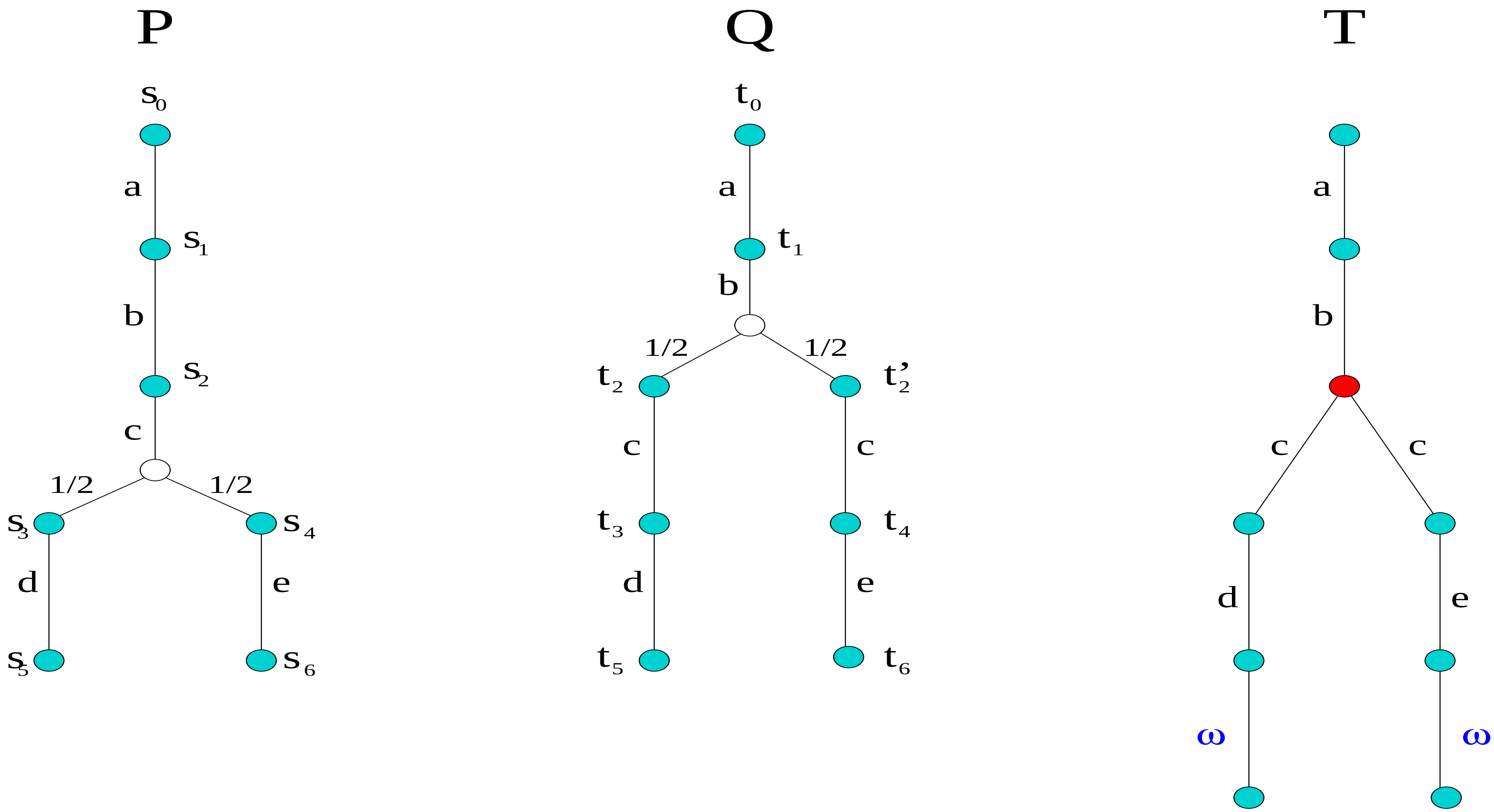
Non-probabilistic vs. probabilistic testing

- Non-probabilistic testing: $\mathcal{O} = \{failure, success\}$
- Probabilistic testing: $\mathcal{O} = [0, 1]$
- Vector based testing: $\mathcal{O} = [0, 1]^n$

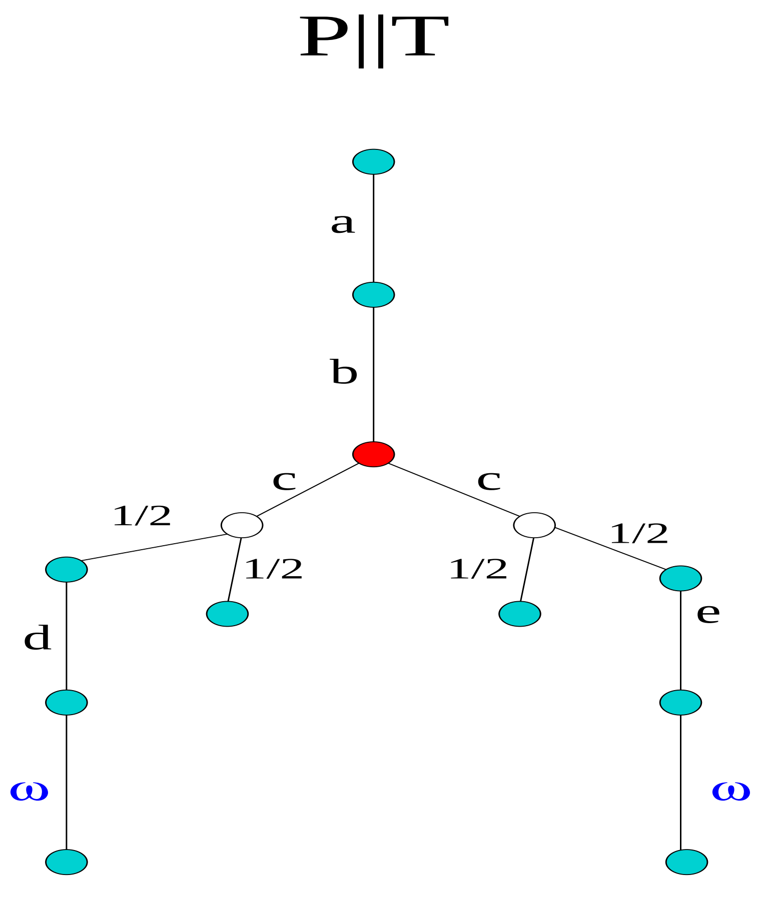
Prop. For closed sets $O_1,\ O_2 \in \mathcal{P}^+_{fin}([0, 1])$ we have

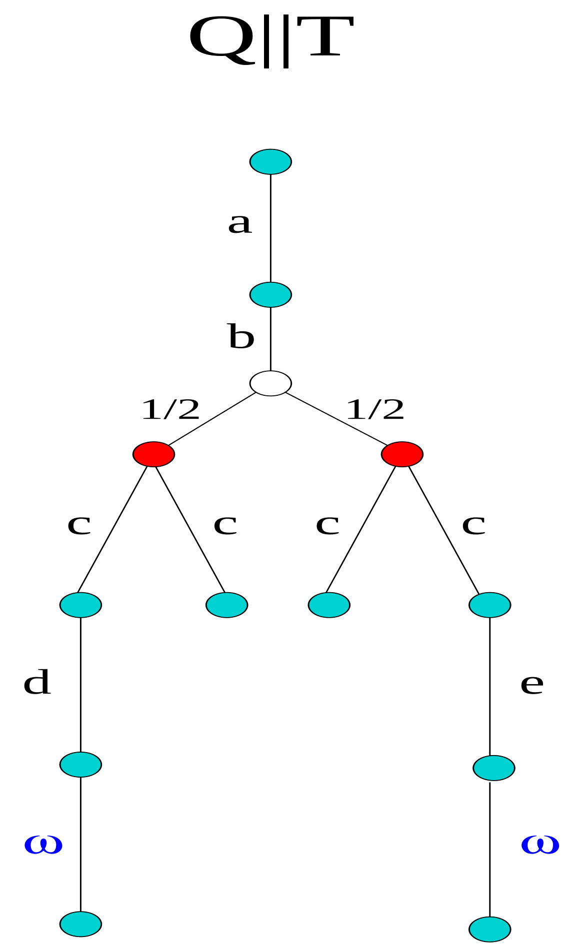
1. $O_1 \leq_{\mathrm{Ho}} O_2$ iff $\max(O_1) \leq \max(O_2)$
2. $O_1 \leq_{\mathrm{Sm}} O_2$ iff $min(O_1) \leq min(O_2)$.

Uni-success testing

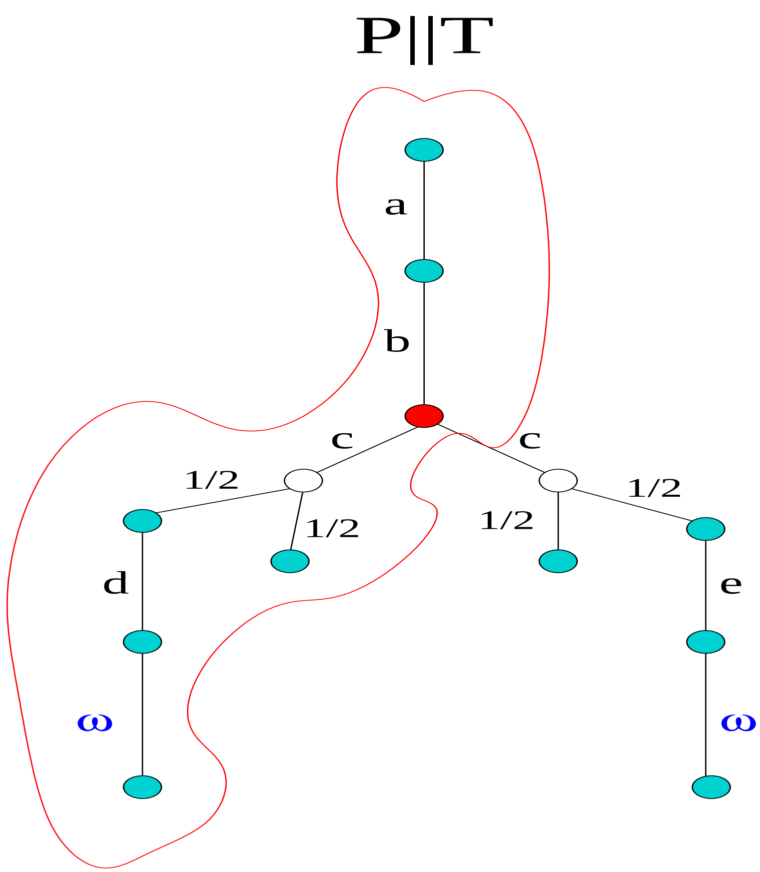


Testing systems

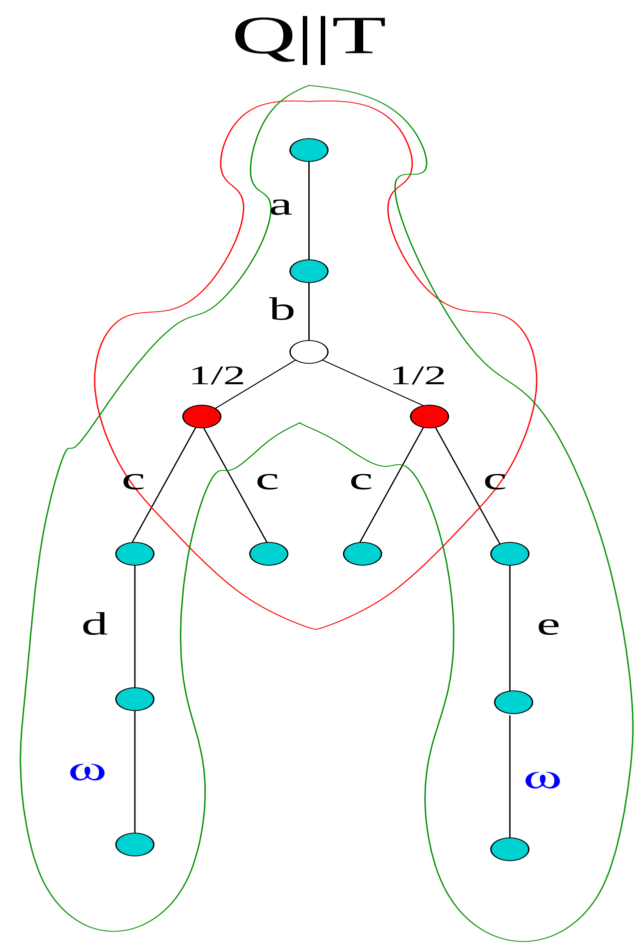




(Static) resolutions



Apply(P||T) = {1/2}



Apply(T, Q) = {0, 1/2, 1}

Uni-success testing preorders

Def.

$$P \sqsubseteq^1_{\texttt{may}} Q \quad \text{if} \quad \forall T : max\{\mathcal{A}(T, P)\} \leq max\{\mathcal{A}(T, Q)\}.$$

$$P \sqsubseteq^1_{\texttt{must}} Q \quad \text{if} \quad \forall T : min\{\mathcal{A}(T, P)\} \leq min\{\mathcal{A}(T, Q)\}$$

E.g. $P \sqsubseteq^1_{\texttt{may}} Q$ and $Q \sqsubseteq^1_{\texttt{must}} P$

Summary

- A notion of probabilistic bisimulation based on a lifting operation
- The lifting is closely related to the Kantorovich metric and network flow problem
- Characterising probabilistic bisimulation via metrics, decision algorithms, and modal logics
- Probabilistic simulations and testing preorders

References

1. C. Baier, B. Engelen, and M. Majster-Cederbaum. Deciding bisimilarity and similarity for probabilistic processes. *Journal of Computer and System Sciences*, 60(1):187-231, 2000.

2. F. van Breugel and J. Worrell. Approximating and computing behavioural distances in probabilistic transition systems. *Theoretical Computer Science* 360:373-385, 2006.

3. J. Desharnais, R. Jagadeesan, V. Gupta, and P. Panangaden.The metric analogue of weak bisimulation for probabilistic processes. In *Proc. LICS'02*.

4. [ESOP'07] Scalar Outcomes Suffice for Finitary Probabilistic Testing.
http://basics.sjtu.edu.cn/∼yuxin/publications/scalar.pdf

5. [LICS'07] Characterising Testing Preorders for Finite Probabilistic Processes (Extended Abstract).
http://basics.sjtu.edu.cn/∼yuxin/publications/lics.pdf

6. [CONCUR'09] Testing Finitary Probabilistic Processes (Extended Abstract).
http://basics.sjtu.edu.cn/∼yuxin/publications/concur09.pdf

7. [QAPL'11] Real-Reward Testing for Probabilistic Processes.
http://basics.sjtu.edu.cn/∼yuxin/publications/qapl2011.pdf

8. [CMURR] Logical, Metric, and Algorithmic Characterisations of Probabilistic Bisimulation.
http://basics.sjtu.edu.cn/∼yuxin/publications/probbisi.pdf