On Böhm Trees and Lévy-Longo Trees in π-calculus

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(from ongoing work with Davide Sangiorgi)

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Subject

Encodings

from λ-calculus (sequential programming)

to π-calculus (concurrent programming)

or variants of name-passing process models.

Benefit

λ in π

1. expressiveness exhibition
2. λ-model in process models
3. full abstraction

Known encodings:

[Milner, 1990] [Sangiorgi, 1993,1994,1995] [Merro and Sangiorgi, 2004] [Cai and Fu, 2011] [Hirschkoff, Madiot, and Sangiorgi, 2012] ...

$$M = N \quad \text{iff} \quad [\![M]\!] \asymp [\![N]\!]$$

Aim of our work

To find general conditions that ensure desired full abstraction of an encoding.

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or Böhm tree (BT) equality

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or Böhm tree (BT) equality

$$M = N \quad \text{iff} \quad [\![M]\!] \asymp [\![N]\!]$$

$=$ is Lévy-Longo tree or Böhm tree equality;

$\asymp$ is a behavioral equivalence in the target model.

Motivation

1. Importance of BT and LT:

(1) Operational semantics of λ-terms

(2) Observational theory in λ (LT and BT equalities)

(3) The local structure of some of influential models of the λ-calculus is the BT equality

(E.g. [Scott & Plotkin's P_ω 1976] [Plotkin's T^ω,1978] [Plotkin & Engeler's D_A, 1981])

2. Proof methods for full abstraction are often tedious:

(1) Operational correspondence

(2) Validity of β rule

(3) Proof technique: Böhm-out, up-to.

(E.g. [Sangiorgi, 1995], [Boudol & Laneve, 1995])

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Organization of this talk

- DEFINITIONS
- THE CONDITIONS
- EXAMPLES
- EXTENSION

Definitions

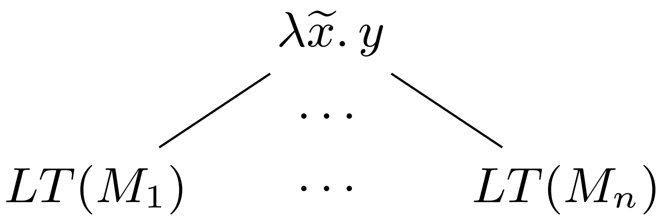
$M \in PO^n$: M has proper order n, i.e. like $\lambda x_1 \ldots x_n . \Omega$.

Definition 1 (Lévy-Longo trees). The *Lévy–Longo Tree* of M, LT(M), is:

(1) $LT(M) = \top$ if $M \in PO^\omega$;

(2) $LT(M) = \lambda x_1 \ldots x_n . \bot$ if $M \in PO^n$, $0 \leqslant n < \omega$;

(3) $LT(M) =$



if $M \to_h^* \lambda \widetilde{x}. y M_1 \ldots M_n$, $n \geqslant 0$.

LT equality: $LT(M) = LT(N)$, i.e. they have the same LTs.

Böhm trees

Böhm trees (BTs):

$$BT(M) = \bot \qquad \text{if } M \in PO^n, 0 \leqslant n \leqslant \omega$$

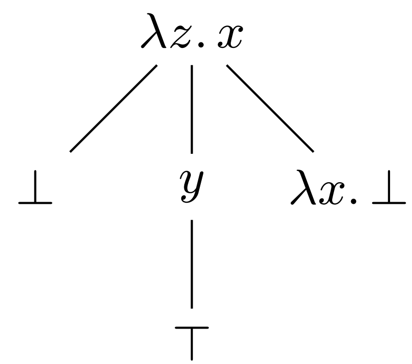
plus (3) of LT.

BT equality: $BT(M) = BT(N)$, i.e. they have the same BTs.

Examples

$M \equiv \lambda z.\, x\Omega(y\Xi)(\lambda x.\, \Omega)$ **(**$\Xi = (\lambda xz.\, xx)(\lambda xz.\, xx)$**)**

$LT(M) =$



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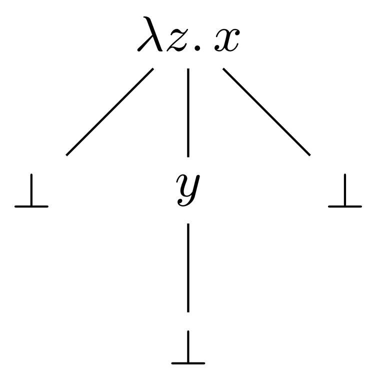

Definition 2 (encoding of the λ-calculus)**.** A mapping from λ-terms to π-agents, and is compositional.

i.e., $[\![\lambda x.\,M]\!] \stackrel{\text{def}}{=} C_\lambda^x[\,[\![M]\!]\,]$ $\qquad [\![M N]\!] \stackrel{\text{def}}{=} C_{\text{app}}[\,[\![M]\!], [\![N]\!]\,]$

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Two kinds of contexts:

- Abstraction context: $C^x_\lambda \stackrel{\text{def}}{=} [\![\lambda x.\, [\cdot]]\!]$
- Variable context: $C^{x,n}_{\text{var}} \stackrel{\text{def}}{=} [\![x[\cdot]_1 \cdots [\cdot]_n]\!]$

Conventions:

- $[\![\ ]\!]$ is an encoding of the λ-calculus into π-calculus
- $\mathcal{V}ar \subseteq \mathcal{N}$, where $\mathcal{N}$ is the set of π-names
- σ stands for name substitution, i.e. mapping on π names
- $\mathcal{C}$ is a set of contexts for π
- $\leq$ is a precongruence and $\asymp$ is a congruence on the agents of π-calculus

Definition 3. $[\![\ ]\!]$ and $\asymp$ are:

- *complete* if
  $LT(M) = LT(N)$ (or $BT(M) = BT(N)$) implies $[\![M]\!] \asymp [\![N]\!]$.
- *sound* if
  $[\![M]\!] \asymp [\![N]\!]$ implies $LT(M) = LT(N)$ (or $BT(M) = BT(N)$).

Full abstraction: soundness & completeness.

Auxiliary definitions (Def.4-6)

Definition 4. $[\![\,]\!]$ and relation $\mathcal{R}$:

- *validate rule* β if $[\![(\lambda x.\,M)N]\!]\ \mathcal{R}\ [\![M\{^N\!/\!_x\}]\!]$.
- *validate rule* α if $[\![\lambda x.\,M]\!]\ \mathcal{R}\ [\![\lambda y.\,(M\{^y\!/\!_x\})]\!]$.

Definition 5. $\mathcal{C}$ is *closed under context composition* if $\forall C \in \mathcal{C}\,.\,\forall D$ (unary context). $D[C] \in \mathcal{C}$.

Definition 6. $\asymp$ has *unique solution of equations up to $\leq$ and the contexts $\mathcal{C}$* if $\forall\mathcal{R}$, it holds that

- If $P\,\mathcal{R}\,Q$ implies
  1. $P \asymp Q$, or
  2. $\exists C \in \mathcal{C}$ with $(1 \leqslant i \leqslant n)$

$$\begin{array}{rcll} P & \geq & C[P_1,\ldots,P_n] & \\ Q & \geq & C[Q_1,\ldots,Q_n] & \\ P_i & \mathcal{R} & Q_i & \\ P_i\sigma & \mathcal{R} & Q_i\sigma & \text{for all } \sigma\text{, if } [\cdot]_i \text{ occurs under an input in } C \end{array}$$

then $\mathcal{R} \subseteq \asymp$.

Intuitively, this definition comes from the proof technique of up-to context and expansion.

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Q & \geq & C[Q_1, \ldots, Q_n] & \\
P_i & \mathcal{R} & Q_i & \\
P_i\sigma & \mathcal{R} & Q_i\sigma & \text{for all } \sigma \text{, if } [\cdot]_i \text{ occurs under an input in } C
\end{array}
$$

  then $\mathcal{R} \subseteq \asymp$.

- Moreover, $\mathcal{R}$ should also be closed under substitution, if the synchronous π-calculus is used.

Intuitively, this definition comes from the proof technique of up-to context and expansion.

The conditions

The conditions for completeness

Theorem 7 (completeness for LT). *$[\![\,]\!]$ and $\asymp$ are complete for LTs, if $\exists \leq, \mathcal{C}$, the conditions below are met.*

1. *the variable contexts of $[\![\,]\!]$ are contained in $\mathcal{C}$;*
2. *either*
   (a) *the abstraction contexts of $[\![\,]\!]$ are contained in $\mathcal{C}$;*
3. $\asymp \supseteq \geq$;
4. *$\asymp$ has unique solution of equations up to $\leq$ and the contexts $\mathcal{C}$;*
5. *$[\![\,]\!]$, $\geq$ validate rules α and β;*
6. *$[\![\,]\!]$ respects substitution, i.e. $[\![M\sigma]\!] \equiv [\![M]\!]\sigma$;*
7. *whenever $M \in PO^0$ then $[\![M]\!] \asymp [\![\Omega]\!]$.*

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2. *either*
   (a) *the abstraction contexts of $[\![\ ]\!]$ are contained in $\mathcal{C}$;*

   or

   (b) *$\mathcal{C}$ is closed under composition and*

   (c) *$M, N \in PO^{\omega}$ imlpies $[\![M]\!] \asymp [\![N]\!]$;*
3. *$\asymp \supseteq \geq$;*
4. *$\asymp$ has unique solution of equations up to $\leq$ and the contexts $\mathcal{C}$;*
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7. *whenever $M \in PO^0$ then $[\![M]\!] \asymp [\![\Omega]\!]$.*

Theorem 8 (completeness for BT). $[\![\;]\!]$ *and* $\asymp$ *are complete for BTs, if* $\exists \leq, \mathcal{C}$*, the conditions below are met.*

1. *the variable contexts of* $[\![\;]\!]$ *are contained in* $\mathcal{C}$*;*
2. *either*
   (a) *the abstraction contexts of* $[\![\;]\!]$ *are contained in* $\mathcal{C}$ *and*
   (b) $[\![\lambda x.\,\Omega]\!] \leq [\![\Omega]\!]$*;*
3. $\asymp \supseteq \geq$*;*
4. $\asymp$ *has unique solution of equations up to* $\leq$ *and the contexts* $\mathcal{C}$*;*
5. $[\![\;]\!]$*,* $\geq$ *validate rules* α *and* β*;*
6. $[\![\;]\!]$ *respects substitution, i.e.* $[\![M\sigma]\!] \equiv [\![M]\!]\sigma$*;*
7. *whenever* $M \in PO^0$ *then* $[\![M]\!] \asymp [\![\Omega]\!]$.

Theorem 8 (completeness for BT). *$[\![\,]\!]$ and $\asymp$ are complete for BTs, if $\exists \leq, \mathcal{C}$, the conditions below are met.*

1. *the variable contexts of $[\![\,]\!]$ are contained in $\mathcal{C}$;*
2. *either*
   (a) *the abstraction contexts of $[\![\,]\!]$ are contained in $\mathcal{C}$ and*
   (b) $[\![\lambda x.\,\Omega]\!] \leq [\![\Omega]\!]$;

   or

   (c) *$\mathcal{C}$ is closed under composition and*
   (d) *$M \in PO^{\omega}$ imlpies $[\![M]\!] \asymp [\![\Omega]\!]$;*
3. $\asymp \supseteq \geq$;
4. *$\asymp$ has unique solution of equations up to $\leq$ and the contexts $\mathcal{C}$;*
5. *$[\![\,]\!]$, $\geq$ validate rules α and β;*
6. *$[\![\,]\!]$ respects substitution, i.e. $[\![M\sigma]\!] \equiv [\![M]\!]\sigma$;*
7. *whenever $M \in PO^0$ then $[\![M]\!] \asymp [\![\Omega]\!]$.*

The conditions for soundness

Definition 9. C: n-hole context.

C has inverse w.r.t. $\geq$, if

$\forall i = 1, \ldots, n$ $\exists D_i$ s.t.

$\forall A_1, \ldots, A_n$, it holds that $D_i[C[\widetilde{A}]] \geq A_i$

Theorem 10 (soundness for LT). *$[\![\,]\!]$ and $\asymp$ are sound for LTs, if $\exists \leq$, the conditions below are satisfied.*

1. *$\geq \subseteq \asymp$ (also $\leq \subseteq \asymp$);*
2. *$[\![\,]\!]$ and $\geq$ validate rules α and β;*
3. *If $M \in PO^0$, then $[\![M]\!] \asymp [\![\Omega]\!] \asymp [\![M]\!]$;*
4. *$[\![\Omega]\!]$, $[\![\lambda x.\, M]\!]$, $[\![x M_1 \cdots M_m]\!]$, $[\![x N_1 \cdots N_n]\!]$, and $[\![y\widetilde{O}]\!]$ are pairwise unequal w.r.t. $\asymp$;*
5. *The abstraction contexts of $[\![\;]\!]$ have inverse with respect to $\geq$;*
6. *The variable contexts of $[\![\;]\!]$ have inverse with respect to $\geq$.*

Theorem 11 (soundness for BT). *$[\![\,]\!]$ and $\asymp$ are sound for BTs, if $\exists \leq$, the conditions below are satisfied.*

1. *$\geq \subseteq \asymp$ (also $\leq \subseteq \asymp$);*

2. *$[\![\,]\!]$ and $\geq$ validate rules α and β;*

3. *If $M \in PO^n$ $(0 \leqslant n \leqslant \omega)$, then $[\![M]\!] \asymp [\![\Omega]\!] \asymp [\![M]\!]$;*

4. *$[\![\Omega]\!]$, $[\![yM_1 \cdots M_m]\!]$, $[\![yN_1 \cdots N_n]\!]$, $[\![z\widetilde{O}]\!]$, and $[\![\lambda x.\, M]\!]$, where $M \notin PO^k$ $(\forall k.\, 0 \leqslant k \leqslant \omega)$, are pairwise unequal w.r.t. $\asymp$;*

5. *The abstraction contexts of $[\![\ ]\!]$ have inverse with respect to $\geq$;*

6. *The variable contexts of $[\![\ ]\!]$ have inverse with respect to $\geq$.*

Examples

Milner's encoding (lazy): $\lambda \rightsquigarrow A\pi$

$$\llbracket \lambda x.\, M \rrbracket \stackrel{\text{def}}{=} (p)\, p(x,q).\, \llbracket M \rrbracket \langle q \rangle$$

$$\llbracket x \rrbracket \stackrel{\text{def}}{=} (p)\, \overline{x}\langle p \rangle$$

$$\llbracket MN \rrbracket \stackrel{\text{def}}{=} (p)\, \boldsymbol{\nu} r, x \left(\llbracket M \rrbracket \langle r \rangle \mid \overline{r}\langle x, p \rangle \mid !x(q).\, \llbracket N \rrbracket \langle q \rangle \right),\ x \text{ fresh.}$$

where $(p)\, P$ is abstraction, and $F\langle p \rangle$ is application.

		$\approx$	$\approx^{asy}$	$\sim_{\text{may}}$	$\sim_{\text{may}}^{\text{asy}}$	$\sim_{must}$
LT	completeness	✓	✓	✓	✓	✓
	soundness	✓	✓	✓		✓
BT	completeness				✓	
	soundness				✓	

Table 1: Results for the encoding

An encoding of strong lazy strategy [HMS12]: $\lambda \leadsto \pi$

$$\llbracket \lambda x.\, M \rrbracket \stackrel{\text{def}}{=} (p)\, \boldsymbol{\nu} x, q\, (\overline{p}\langle x, q\rangle \mid \llbracket M \rrbracket \langle q\rangle)$$

$$\llbracket x \rrbracket \stackrel{\text{def}}{=} (p)\, (x(p').\, (p' \rhd p))$$

$$\llbracket MN \rrbracket \stackrel{\text{def}}{=} (p)\, \boldsymbol{\nu} q, r\, (\llbracket M \rrbracket \langle q\rangle \mid q(x, p').\, (p' \rhd p \mid !\overline{x}\langle r\rangle.\, \llbracket N \rrbracket \langle r\rangle)) \qquad x \text{ fresh}$$

where $r \rhd q \stackrel{\text{def}}{=} !r(y, h).\, \overline{q}\langle y, h\rangle$; $\rhd$ has the most precedence.

		$\approx^c$	$\sim^c_{may}$	$\sim^c_{must}$
LT	completeness	✓	✓	✓
	soundness	✓	✓	
BT	completeness			✓
	soundness			✓

Table 2: Results for the encoding

Extension

Question

How can we tune the observational theory to obtain BT rather than LT?

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Basic idea: assign only writable type to the access point p of the encoding, so that pairs like $[\![\Omega]\!]$ and $[\![\lambda x.\,\Omega]\!]$ would become equal.

		$\approx_t$	$\approx_t^a$
LT	completeness	✓	✓
	soundness	✓	
BT	completeness		✓
	soundness	✓	✓

Table 3: More results for Milner's encoding under types

Conclusion

What's done:

- general conditions for encodings of λ into π
- case study of the conditions
- extension using types (obtain BT without ξ rule)

Future directions:

- other (process) models, e.g. higher-order, ambients.
- BT with η rule (more technique needed)

Thank you