Deciding Kleene Algebra with converse is PSPACE-*complete*

Talk at the PACE meeting

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ENS de Lyon

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Introduction

Kleene Algebra (i) : Abstraction for proving the equivalence of regular expressions.

(i). Conway, J. H. (1971). *Regular algebra and finite machines.* Chapman and Hall Mathematics Series

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What if we add a *converse* operation to regular expressions ?

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Introduction

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Plan

1. Introduction
2. From Kleene Algebra with Converse to regular languages
    - Kleene Algebra wih converse
    - Reduction to an automaton problem
3. Closure of an automaton
4. The PSpace algorithm.

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Regular expressions with converse

Regular expressions with converse over X

Let X be a finite set, the set of regular expressions over X (written $\mathcal{R}eg_X{}^\vee$) are obtained with the grammar :

$$e, f ::= \mathbb{0} | \mathbb{1} | x \in X | e + f | e \cdot f | e^* | e^\vee$$

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A relational interpretation of regular expressions with converse over X can be specified by a domain S and a map

$$\sigma : X \longrightarrow \mathcal{P}\left(S^2\right)$$

We will write

$$\hat{\sigma} : \mathcal{R}eg_X{}^{\vee} \longrightarrow \mathcal{P}\left(S^2\right)$$

for the unique morphism equal to σ on X.

Relational equivalence

For $e, f \in \mathcal{R}eg_X{}^{\vee}$:

$$e \equiv_{\mathcal{R}el^{\vee}} f$$

means that

$$\forall S, \forall \sigma : X \to \mathcal{P}\left(S^2\right), \hat{\sigma}(e) = \hat{\sigma}(f).$$

A hint from the equational theory.

$$(a+b)^{\vee} = a^{\vee} + b^{\vee} \tag{1}$$

$$(a \cdot b)^{\vee} = b^{\vee} \cdot a^{\vee} \tag{2}$$

$$(a^*)^{\vee} = (a^{\vee})^* \tag{3}$$

$$a^{\vee\vee} = a \tag{4}$$

$$a \leqslant aa^{\vee}a \tag{5}$$

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From ${\mathcal{R}eg_X}^{\vee}$ to $\mathcal{R}eg_{\mathbf{X}}$

Let X be a finite alphabet. For $e \in \mathcal{R}eg_X$, we write $[\![e]\!] \subseteq X^*$ for the *language denoted by e*.

- $X' := \{x' \mid x \in X\}$ is a disjoint copy of X,
- and $\mathbf{X} := X \cup X'$.

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1. We see equations (1)-(4) as rewriting rules :

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\end{array}$$

2. We substitute $x^\vee$ with x' in the result. We get $\mathbf{e} \in \mathcal{R}eg_{\mathbf{X}}$.

Reduction relation

$$a \leqslant a a^{\vee} a$$

$\overline{w}$

For a word $w \in \mathbf{X}^*$, we define inductively $\overline{w}$:

$$\begin{array}{rrl|rl} \forall x \in X, & \overline{x} := x' & & \overline{\epsilon} := \epsilon \\ \forall x' \in X', & \overline{x'} := x & & \overline{wx} := \overline{x}\ \overline{w} \end{array}$$

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$u \rightsquigarrow v$

$$\frac{}{u_1 \cdot w\overline{w}w \cdot u_2 \quad \rightsquigarrow \quad u_1 \cdot w \cdot u_2}$$

Example :

$abbabb'a'abbaa'$

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Closure

$\mathcal{cl}(L)$

$$\mathcal{cl}(L) := \{v \mid \exists u \in L : u \rightsquigarrow^{*} v\}$$

Closure

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$$cl(L) := \{v \mid \exists u \in L : u \rightsquigarrow^* v\}$$

Theorem [a]

---

[a] Bloom, S. L., Ésik, Z., and Stefanescu, G. (1995). Notes on equational theories of relations.
Algebra Universalis, 33(1) :98–126

$$e \equiv_{\mathcal{R}el^{\vee}} f \quad \Leftrightarrow \quad cl(\llbracket \mathbf{e} \rrbracket) = cl(\llbracket \mathbf{f} \rrbracket)$$

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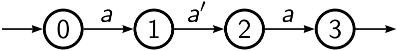
Problem

Input : an automaton $\mathscr{A}$

Output : an automaton $\mathscr{A}'$ such that $L(\mathscr{A}') = \mathscr{cl}(L(\mathscr{A}))$.

Intuition



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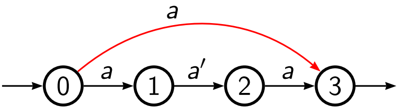

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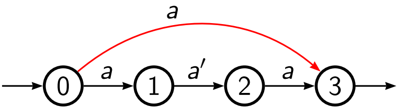

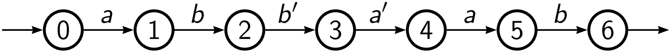

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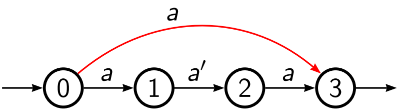

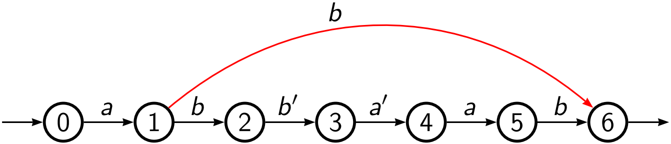

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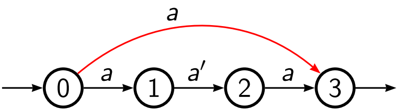

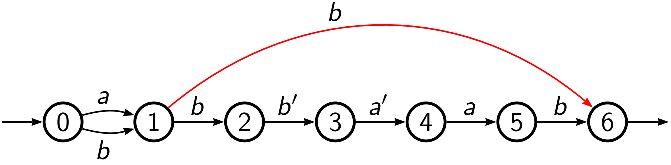

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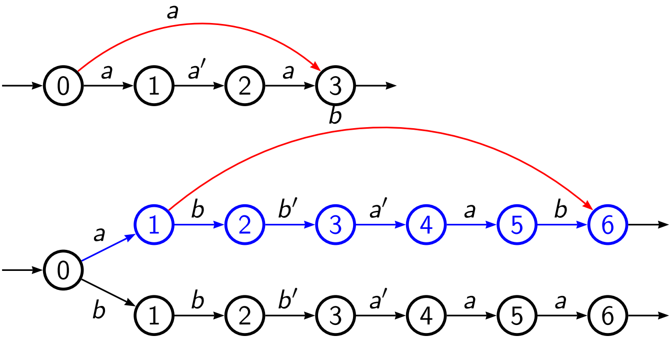

General idea

- Bloom, Ésik and Stefanescu give a construction, using the transitions monoid of the initial automaton, building a deterministic automaton with size $2^{2^{n^2}}$ (if the initial automaton has size n).

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$$\text{With} : \exists u_2 \in \text{suffixes}(ux) : w \rightsquigarrow^* \overline{u}_2 u_2$$

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$\Gamma(w)$

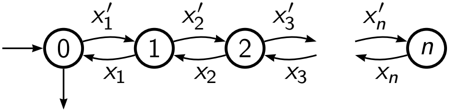
Definition : $\Gamma(w)$

$$\Gamma(\epsilon) = \{\epsilon\}$$
$$\Gamma(wx) = (\{x'\} \cdot \Gamma(w) \cdot \{x\})^*$$

Lemma

$$u \in \Gamma(w) \Leftrightarrow \exists v \in \text{suffixes}(w) : u \rightsquigarrow^* \overline{v}v$$

$\Gamma(x_n \cdots x_1)$ is recognised by the automaton :



$\gamma(w)$

Consider an automaton $\mathscr{A} = \langle Q, A, I, T, \Delta\rangle$, we write $\Delta_x \coloneqq \{(p,q) \mid p \xrightarrow{x} q \in \Delta\}$.

Definition : $\gamma(w)$

$$\gamma(\epsilon) = \mathrm{Id}_Q$$
$$\gamma(wx) = (\Delta_{x'} \cdot \gamma(w) \cdot \Delta_x)^*$$

Lemma

$$\begin{aligned}(p,q) \in \gamma(w) &\Leftrightarrow \exists u \in \Gamma(w) : p \xrightarrow{u} q\\ &\Leftrightarrow \exists u : \exists v \in \mathrm{suffixes}(w) : p \xrightarrow{u} q \wedge u \rightsquigarrow^* \overline{v}v\end{aligned}$$

Histories

The set of histories is $G \coloneqq \left\{r \in \mathcal{P}\left(Q^2\right) \mid \exists w \in \mathbf{X}^* : r = \gamma(w)\right\}$.

Closure Automaton

$\mathcal{cl}(\mathscr{A})$

$\mathcal{cl}(\mathscr{A}) := \langle Q \times G, \mathbf{X}, I \times \gamma(\epsilon), F \times G, \Delta' \rangle$ with transitions Δ' :

$$(q_1, \gamma(w)) \xrightarrow{x}_{\mathcal{cl}(\mathscr{A})} (q_2, \gamma(wx)) \text{ if } (q_1, q_2) \in \Delta_x \circ \gamma(wx)$$

Theorem

$$L(\mathcal{cl}(\mathscr{A})) = \mathcal{cl}(L(\mathscr{A}))$$

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$$(p, \gamma(u)) \xrightarrow{x} (q, \gamma(ux)) \triangleq \begin{array}{l} \exists r \in Q \\ \exists w \in \Gamma(ux) \end{array} \;:\; p \xrightarrow{x} r \xrightarrow{w} q$$

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Size

$$\Delta' : \{((q_1, \gamma(w)), x, (q_2, \gamma(wx))) \mid (q_1, q_2) \in \Delta_x \circ \gamma(wx)\}$$

We can see that this construction produces a non-deterministic automaton of size at most $n \times 2^{n\times(n-1)}$.

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Futhurmore, it can be easily determinized :

$$\delta' : ((Q_1, \gamma(w)), x) \mapsto (Q_1 \cdot (\Delta_x \circ \gamma(wx))\,, \gamma(wx))$$

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This deterministic automaton has at most $2^n \times 2^{n \times (n-1)} = 2^{n^2}$ states, which is significantely smaller than $2^{2^{n^2}}$, the size of the automaton from the original construction.

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Automaton equivalence

Let $\mathscr{A}$ and $\mathscr{B}$ be two deterministic automata over some alphabet Σ.

Theorem

$L(\mathscr{A}) \neq L(\mathscr{B}) \Leftrightarrow \exists w \in (L(\mathscr{A}) \setminus L(\mathscr{B})) \cup (L(\mathscr{B}) \setminus L(\mathscr{A})) : |w| \leqslant |\mathscr{A}| \times |\mathscr{B}|.$

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```
input  : $\mathscr{A}_1 = \langle Q_1, \Sigma, i_1, T_1, \delta_1 \rangle$
input  : $\mathscr{A}_2 = \langle Q_2, \Sigma, i_2, T_2, \delta_2 \rangle$
output: A Boolean, saying whether or not $\mathscr{A}_1$ and $\mathscr{A}_2$ recognise the same language.

$N \leftarrow (|Q_1| \times |Q_2|)$ ;
$(p_1, p_2) \leftarrow (i_1, i_2)$ ;
while $N > 0$ do
    $N \leftarrow N - 1$;                                   /* $N$ bounds the recursion depth */
    $f_1 \leftarrow$ is_in($p_1, T_1$);
    $f_2 \leftarrow$ is_in($p_2, T_2$);
    if $f_1 = f_2$ then
        $\boxed{x \leftarrow \texttt{random}(\Sigma)\ ;}$                /* Non-deterministic choice */
        $(p_1, p_2) \leftarrow (\delta_1(p_1, x), \delta_2(p_2, x))$ ;
    else
        return false;           /* A difference appeared for some word, $L(\mathscr{A}_1) \neq L(\mathscr{A}_2)$ */
    end

end
return true;                    /* There was no difference, $L(\mathscr{A}_1) = L(\mathscr{A}_2)$ */
```

Automaton equivalence

Let $\mathscr{A}$ and $\mathscr{B}$ be two deterministic automata over some alphabet Σ.

Theorem

$$L(\mathscr{A}) \neq L(\mathscr{B}) \Leftrightarrow \exists w \in (L(\mathscr{A}) \setminus L(\mathscr{B})) \cup (L(\mathscr{B}) \setminus L(\mathscr{A})) : |w| \leqslant |\mathscr{A}| \times |\mathscr{B}|.$$

input : $\mathscr{A}_1 = \langle Q_1, \Sigma, i_1, T_1, \delta_1 \rangle$
input : $\mathscr{A}_2 = \langle Q_2, \Sigma, i_2, T_2, \delta_2 \rangle$
output: A Boolean, saying whether or not $\mathscr{A}_1$ and $\mathscr{A}_2$ recognise the same language.

```
$\boxed{N \leftarrow (|Q_1| \times |Q_2|)}$ ;
$(p_1, p_2) \leftarrow (i_1, i_2)$ ;
while $N > 0$ do
    $N \leftarrow N - 1$;                              /* $N$ bounds the recursion depth */
    $f_1 \leftarrow$ is_in($p_1$, $T_1$);
    $f_2 \leftarrow$ is_in($p_2$, $T_2$);
    if $f_1 = f_2$ then
        $x \leftarrow$random($\Sigma$) ;                /* Non-deterministic choice */
        $(p_1, p_2) \leftarrow (\delta_1(p_1, x), \delta_2(p_2, x))$ ;
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    $f_1 \leftarrow$ is_in($p_1, T_1$);
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    if $f_1 = f_2$ then
        $\boxed{x \leftarrow \texttt{random}(\Sigma)\ ;}$          /* Non-deterministic choice */
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```

A PSPACE algorithm for KAC

```
input  : Two regular expressions with converse $e, f \in \mathcal{R}eg_X{}^{\vee}$
output: A Boolean, saying whether or not $\text{KAC} \vdash e = f$.
$\mathscr{A}_1 = \langle Q_1, \mathbf{X}, I_1, T_1, \Delta_1 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{e}]\!]$ ;
$\mathscr{A}_2 = \langle Q_2, \mathbf{X}, I_2, T_2, \Delta_2 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{f}]\!]$ ;
$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$ ;
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$ ;
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty($P_1 \cap T_1$);
    $f_2 \leftarrow$ is_empty($P_2 \cap T_2$);
    if $f_1 = f_2$ then
        $x \leftarrow$random($\mathbf{X}$);
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^{\star}, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^{\star})$ ;
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;
    else
        return false
    end
end
return true
```

A PSpace algorithm for KAC

Let's write n and m for the sizes of e and f.

input : Two regular expressions with converse $e, f \in \mathcal{R}eg_X{}^\vee$
output: A Boolean, saying whether or not KAC $\vdash e = f$.

```
$\mathscr{A}_1 = \langle Q_1, \mathbf{X}, I_1, T_1, \Delta_1 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{e}]\!]$;
$\mathscr{A}_2 = \langle Q_2, \mathbf{X}, I_2, T_2, \Delta_2 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{f}]\!]$;
$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$;
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$;
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty($P_1 \cap T_1$);
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    if $f_1 = f_2$ then
        $x \leftarrow$random($\mathbf{X}$);
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$;
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$;
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$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$ ;
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$ ;
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty$(P_1 \cap T_1)$;
    $f_2 \leftarrow$ is_empty$(P_2 \cap T_2)$;
    if $f_1 = f_2$ then
        $x \leftarrow$random$(\mathbf{X})$;
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$ ;
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$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$;
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$;
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```

$\mathcal{O}(n+m)$

A PSpace algorithm for KAC

Let's write n and m for the sizes of e and f.

```
input  : Two regular expressions with converse $e, f \in \mathcal{R}eg_X^{\vee}$
output: A Boolean, saying whether or not KAC $\vdash e = f$.
$\mathscr{A}_1 = \langle Q_1, \mathbf{X}, I_1, T_1, \Delta_1 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{e}]\!]$ ;  $\leftarrow \boxed{\mathcal{O}(n+m)}$
$\mathscr{A}_2 = \langle Q_2, \mathbf{X}, I_2, T_2, \Delta_2 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{f}]\!]$ ;
$\boxed{N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2});}$
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$ ;
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty($P_1 \cap T_1$);
    $f_2 \leftarrow$ is_empty($P_2 \cap T_2$);
    if $f_1 = f_2$ then
        $x \leftarrow$random($\mathbf{X}$);
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$ ;
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$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$ ;
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```

$\mathcal{O}\left(n+m\right)$

$\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2+m^2\right)$

A PSpace algorithm for KAC

Let's write n and m for the sizes of e and f.

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$\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2 + m^2\right)$

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output : A Boolean, saying whether or not $\mathrm{KAC} \vdash e = f$.
$\mathscr{A}_1 = \langle Q_1, \mathbf{X}, I_1, T_1, \Delta_1 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{e}]\!]$;
$\mathscr{A}_2 = \langle Q_2, \mathbf{X}, I_2, T_2, \Delta_2 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{f}]\!]$;
$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$;
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$;
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty($P_1 \cap T_1$);
    $f_2 \leftarrow$ is_empty($P_2 \cap T_2$);
    if $f_1 = f_2$ then
        $x \leftarrow$random($\mathbf{X}$);
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$;
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$;
    else
        return false
    end
end
return true
```

$\mathcal{O}\left(n + m\right)$

$\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2 + m^2\right)$

$\sim log(n) + log(m) + n^2 + m^2 \sim \mathcal{O}\left(n^2 + m^2\right)$

A PSpace algorithm for KAC

Let's write n and m for the sizes of e and f.

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        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;
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        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$ ;
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;
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$\sim log(n) + log(m) + n^2 + m^2 \sim \mathcal{O}\left(n^2 + m^2\right)$

$\mathcal{O}(log(n))$

A PSPACE algorithm for KAC

Let's write n and m for the sizes of e and f.

input : Two regular expressions with converse $e, f \in \mathcal{R}eg_X{}^{\vee}$
output: A Boolean, saying whether or not KAC $\vdash e = f$.

```
$\mathscr{A}_1 = \langle Q_1, \mathbf{X}, I_1, T_1, \Delta_1 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{e}]\!]$ ;    ← $\mathcal{O}(n+m)$
$\mathscr{A}_2 = \langle Q_2, \mathbf{X}, I_2, T_2, \Delta_2 \rangle \leftarrow$ Glushkov' automaton recognising $[\![\mathbf{f}]\!]$ ;
$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$ ;    ← $\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2+m^2\right)$
$((P_1, R_1), (P_2, R_2)) \leftarrow ((I_1, \mathrm{Id}_{Q_1}), (I_2, \mathrm{Id}_{Q_1}))$ ;    ← $\sim log(n) + log(m) + n^2 + m^2 \sim \mathcal{O}\left(n^2+m^2\right)$
while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty($P_1 \cap T_1$);
    $f_2 \leftarrow$ is_empty($P_2 \cap T_2$);    ← $\mathcal{O}(log(n))$
    if $f_1 = f_2$ then
        $x \leftarrow$random($\mathbf{X}$);
        $\boxed{(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star) ;}$
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;
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while $N > 0$ do
    $N \leftarrow N - 1$;
    $f_1 \leftarrow$ is_empty$(P_1 \cap T_1)$;
    $f_2 \leftarrow$ is_empty$(P_2 \cap T_2)$;
    if $f_1 = f_2$ then
        $x \leftarrow$random$(\mathbf{X})$;
        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^\star, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^\star)$ ;
        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;
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```

$\mathcal{O}\left(n+m\right)$

$\sim log\left(2^{n^2} \times 2^{m^2}\right) \sim \mathcal{O}\left(n^2+m^2\right)$

$\sim log(n) + log(m) + n^2 + m^2 \sim \mathcal{O}\left(n^2+m^2\right)$

$\mathcal{O}\left(log(n)\right)$

n^2

A PSPACE algorithm for KAC

Let's write n and m for the sizes of e and f.

```
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output: A Boolean, saying whether or not $\mathrm{KAC} \vdash e = f$.
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$N \leftarrow (2^{(|e|+1)^2} \times 2^{(|f|+1)^2})$;
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    if $f_1 = f_2$ then
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        $(R_1, R_2) \leftarrow ((\Delta_1(x') \circ R_1 \circ \Delta_1(x))^{\star}, (\Delta_2(x') \circ R_2 \circ \Delta_2(x))^{\star})$;
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$\mathcal{O}(log(n))$

$2 \times m^2$

A PSpace algorithm for KAC

Let's write n and m for the sizes of e and f.

```
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$((P_1,R_1),(P_2,R_2)) \leftarrow ((I_1,\mathrm{Id}_{Q_1}),(I_2,\mathrm{Id}_{Q_1}))$ ;      $\sim log(n) + log(m) + n^2 + m^2 \sim \mathcal{O}\left(n^2+m^2\right)$
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        $\boxed{(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))\,;}$   $\mathcal{O}\left(n^2+m^2\right)$
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        $(P_1, P_2) \leftarrow (P_1 \cdot (\Delta_1(x) \circ R_1), P_2 \cdot (\Delta_2(x) \circ R_2))$ ;    $\mathcal{O}\left(n^2 + m^2\right)$
    else
        return false
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```

So we get a space complexity $\mathcal{O}\left(n^2 + m^2\right)$.

Plan

1. Introduction
2. From Kleene Algebra with Converse to regular languages
   - Kleene Algebra wih converse
   - Reduction to an automaton problem
3. Closure of an automaton
4. The PSPACE algorithm.