Probabilistic Bisimilarity Revisited

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Outline

1. Preliminaries
2. Probabilistic bisimulation and simulation
3. A modal characterisation of probabilistic bisimulation

Preliminaries

Probability distributions

- A (discrete) probability distribution over a countable set S is a function $\Delta : S \rightarrow [0,1]$ s.t. $\sum_{s \in S} \Delta(s) = 1$
- The support of Δ: $\lceil \Delta \rceil := \{s \in S | \Delta(s) > 0\}$
- $\mathcal{D}(S)$: the set of all distributions over S
- $\overline{s}$: the point distribution $\overline{s}(s) = 1$
- Given distributions $\Delta_1, ..., \Delta_n$, we form their linear combination $\sum_{i \in 1..n} p_i \cdot \Delta_i$, where $\forall i : p_i > 0$ and $\sum_{i \in 1..n} p_i = 1$.

Probabilistic labelled transition systems

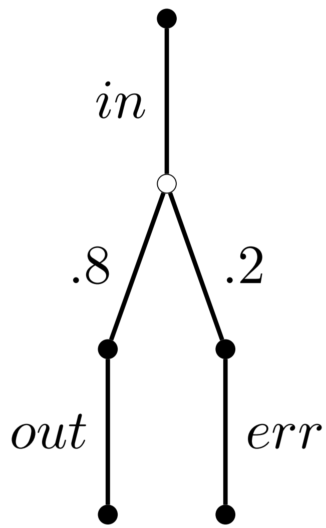
Def. A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, Act, \rightarrow \rangle$, where

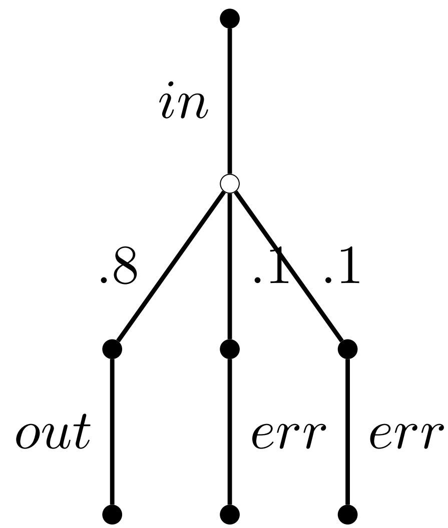
1. S is a set of states
2. Act is a set of actions
3. $\rightarrow \;\subseteq\; S \times Act \times \mathcal{D}(S)$.

We usually write $s \xrightarrow{\alpha} \Delta$ in place of $(s, \alpha, \Delta) \in \rightarrow$. An LTS may be viewed as a degenerate pLTS that only uses point distributions.

A pLTS is reactive if $\rightarrow$ is a *function* from $S \times Act$ to $\mathcal{D}(S)$.

Example





Lifting relations

Def. Let $\mathcal{R} \subseteq S \times T$ be a relation between sets S and T. Then $\mathcal{R}^\dagger \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is the smallest relation that satisfies:

1. $s \; \mathcal{R} \; t$ implies $\overline{s} \; \mathcal{R}^\dagger \; \overline{t}$
2. $\Delta_i \; \mathcal{R}^\dagger \; \Theta_i$ implies $(\sum_{i\in I} p_i \cdot \Delta_i) \; \mathcal{R}^\dagger \; (\sum_{i\in I} p_i \cdot \Theta_i)$ for any $p_i \in [0,1]$ with $\sum_{i\in I} p_i = 1$.

Alternative ways of lifting (1/2)

Prop. $\Delta \, \mathcal{R}^{\dagger} \, \Theta$ if and only if

1. $\Delta = \sum_{i \in I} p_i \cdot \overline{s_i}$, where I is a countable index set and $\sum_{i \in I} p_i = 1$
2. For each $i \in I$ there is a state t_i such that $s_i \, \mathcal{R} \, t_i$
3. $\Theta = \sum_{i \in I} p_i \cdot \overline{t_i}$.

Alternative ways of lifting (2/2)

Prop. Let Δ, Θ be distributions over S and $\mathcal{R}$ be an equivalence relation. Then

$$\Delta \; \mathcal{R}^{\dagger} \; \Theta \quad \text{iff} \quad \forall C \in S/\mathcal{R} : \Delta(C) = \Theta(C)$$

where $\Delta(C) = \sum_{s \in C} \Delta(s)$.

A useful property

Lem. Let $\Delta, \Theta \in \mathcal{D}(S)$ and R be a preorder on S. If $\Delta \; R^\dagger \; \Theta$ then $\Delta(A) \leq \Theta(R(A))$ for each set $A \subseteq S$.

Cor. Let $\Delta, \Theta \in \mathcal{D}(S)$ and R be a preorder on S. If $\Delta \; R^\dagger \; \Theta$ then $\Delta(A) \leq \Theta(A)$ for each R-closed set $A \subseteq S$.

NB: $R(A) = \{t \mid \exists s \in A, \; s \; R \; t\}$. A set A is R-closed if $R(A) \subseteq A$.

The key lemma

Lem. Let R be a preorder on a set S and $\Delta, \Theta \in \mathcal{D}(S)$. If $\Delta \; R^{\dagger} \; \Theta$ and $\Theta \; R^{\dagger} \; \Delta$ then $\Delta(C) = \Theta(C)$ for all equivalence classes C with respect to the kernel $R \cap R^{-1}$ of R.

C. Baier's proof relies on the machinery of DCPOs.

We give an elementary proof with basic concepts of set thoery.

The key lemma

Lem. Let R be a preorder on a set S and $\Delta, \Theta \in \mathcal{D}(S)$. If $\Delta\ R^\dagger\ \Theta$ and $\Theta\ R^\dagger\ \Delta$ then $\Delta(C) = \Theta(C)$ for all equivalence classes C with respect to the kernel $R \cap R^{-1}$ of R.

Proof. Let $\equiv = R \cap R^{-1}$ and $[s]_\equiv$ the equivalence class that contains s.

$$\begin{array}{rcl} R(s) & = & \{t \in S \mid s\ R\ t\} \\ & = & \{t \in S \mid s\ R\ t\ \wedge\ t\ R\ s\} \uplus \{t \in S \mid s\ R\ t\ \wedge\ t\ \not R\ s\} \\ & = & [s]_\equiv\ \uplus\ A_s \end{array}$$

where $\uplus$ stands for a disjoint union.

$$\Delta(R(s)) = \Delta([s]_\equiv) + \Delta(A_s) \qquad \text{and} \qquad \Theta(R(s)) = \Theta([s]_\equiv) + \Theta(A_s)$$

Check that both $R(s)$ and A_s are R-closed sets. Since $\Delta\ R^\dagger\ \Theta$ and $\Theta\ R^\dagger\ \Delta$, use the last corollary and obtain $\Delta(R(s)) = \Theta(R(s))$. Similarly, $\Delta(A_s) = \Theta(A_s)$ It follows that $\Delta([s]_\equiv) = \Theta([s]_\equiv)$.

Probabilistic bisimulation and simulation

Bisimulation

Def. A binary relation $\mathcal{R} \subseteq S \times S$ is a simulation if whenever $s \,\mathcal{R}\, t$:

- if $s \xrightarrow{a} \Delta$, there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta \;\mathcal{R}^{\dagger}\; \Theta$.

The relation $\mathcal{R}$ is a bisimulation if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are simulations. Bisimilarity, written $\sim$, is the union of all bisimulations. The largest simulation is similarity, written $\prec$. The kernel of probabilistic similarity, i.e $\prec \cap \prec^{-1}$, is called simulation equivalence, denoted by $\asymp$.

Simulation equivalence

Thm. For reactive pLTSs, simulation equivalence coincides with bisimilarity.

Proof. Show that $\asymp$ is a bisimulation. Suppose $s \asymp t$. If $s \xrightarrow{a} \Delta$ then $t \xrightarrow{a} \Theta$ for some Θ with $\Delta \prec^{\dagger} \Theta$. For reactive pLTSs, $t \xrightarrow{a} \Theta$ must be matched by $s \xrightarrow{a} \Delta$ and $\Theta \prec^{\dagger} \Delta$. From the previous lemma, $\Delta(C) = \Theta(C)$ for any $C \in S/\asymp$.

A model characterisation of bisimulation

The logic

The language $\mathcal{L}$ of formulas:

$$\varphi ::= \quad \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_p \varphi.$$

Modal characterisation for the continuous case given by Panagaden et al.

We will see the concrete case can be much simplified.

Semantics

- $s \models \top$ always;
- $s \models \varphi_1 \wedge \varphi_2$, if $s \models \varphi_1$ and $s \models \varphi_2$;
- $s \models \langle a \rangle_p \varphi$, if $s \xrightarrow{a} \Delta$ and $\exists A \subseteq S.\ (\forall s' \in A.\ s' \models \varphi) \wedge (\Delta(A) \geq p)$.

Let $[\![\varphi]\!] = \{s \in S \mid s \models \varphi\}$. Then $s \models \langle a \rangle_p \varphi$ iff $s \xrightarrow{a} \Delta$ and $\Delta([\![\varphi]\!]) \geq p$.

Logical equivalence

Let $s =^{\mathcal{L}} t$ if $s \models \varphi \Leftrightarrow t \models \varphi$ for all $\varphi \in \mathcal{L}$.

Lem. Given a reactive pLTS $(S, A, \longrightarrow)$ and two states $s, t \in S$, if $s =^{\mathcal{L}} t$ and $s \xrightarrow{a} \Delta$, then some Θ exists with $t \xrightarrow{a} \Theta$, and for any formula $\psi \in \mathcal{L}$ we have $\Delta(\llbracket\psi\rrbracket) = \Theta(\llbracket\psi\rrbracket)$.

The π-λ theorem

Let $\mathcal{P}$ be a family of subsets of a set X. $\mathcal{P}$ is a π-class if is closed under finite intersection; $\mathcal{P}$ is a λ-class if it is closed under complementations and countable disjoint unions.

Thm. If $\mathcal{P}$ is a π-class, then $\sigma(\mathcal{P})$ is the smallest λ-class containing $\mathcal{P}$, where $\sigma(\mathcal{P})$ is a σ-algebra containing $\mathcal{P}$.

An application of the π-λ theorem

Prop. Let $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}\}$. For any $\Delta, \Theta \in \mathcal{D}(S)$, if $\Delta(A) = \Theta(A)$ for any $A \in \mathcal{A}_0$, then $\Delta(B) = \Theta(B)$ for any $B \in \sigma(\mathcal{A}_0)$.

An application of the π-λ theorem

Prop. Let $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}\}$. For any $\Delta, \Theta \in \mathcal{D}(S)$, if $\Delta(A) = \Theta(A)$ for any $A \in \mathcal{A}_0$, then $\Delta(B) = \Theta(B)$ for any $B \in \sigma(\mathcal{A}_0)$.

Proof. Let

$$\mathcal{P} = \{A \in \sigma(\mathcal{A}_0) \mid \Delta(A) = \Theta(A)\}.$$

$\mathcal{P}$ is closed under countable disjoint unions because probability distributions are σ-additive. $\mathcal{P}$ is closed under complementation because if $A \in \mathcal{P}$ then $\Delta(S \backslash A) = \Delta(S) - \Delta(A) = \Theta(S) - \Theta(A) = \Theta(S \backslash A)$. Thus $\mathcal{P}$ is a λ-class. Note that $\mathcal{A}_0$ is a π-class because $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$. Since $\mathcal{A}_0 \subseteq \mathcal{P}$, we apply the π-λ Theorem to obtain that $\sigma(\mathcal{A}_0) \subseteq \mathcal{P} \subseteq \sigma(\mathcal{A}_0)$, i.e. $\sigma(\mathcal{A}_0) = \mathcal{P}$.

Completeness of the logic

Lem. Given the logic $\mathcal{L}$, and let $(S, A, \longrightarrow)$ be a reactive pLTS. Then for any two states $s, t \in S$, $s \sim t$ iff $s =^{\mathcal{L}} t$.

Completeness of the logic

Lem. Given the logic $\mathcal{L}$, and let $(S, A, \longrightarrow)$ be a reactive pLTS. Then for any two states $s, t \in S$, $s \sim t$ iff $s =^{\mathcal{L}} t$.

Proof. For any $u \in S$ the equivalence class in $S/_{=^{\mathcal{L}}}$ that contains u is

$$[u] = \bigcap\{[\![\varphi]\!] \mid u \models \varphi\} \cap \bigcap\{S \backslash [\![\varphi]\!] \mid u \not\models \varphi\}.$$

Here only countable intersections are used because the set of all the formulas in the logic $\mathcal{L}$ is countable. Let $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}\}$. Then each equivalence class of $S/_{=^{\mathcal{L}}}$ is a member of $\sigma(\mathcal{A}_0)$.

$s =^{\mathcal{L}} t$ and $s \xrightarrow{a} \Delta$ implies that some Θ exists with $t \xrightarrow{a} \Theta$ and for any $\varphi \in \mathcal{L}$, $\Delta([\![\varphi]\!]) = \Theta([\![\varphi]\!])$. By the last proposition, $\Delta([u]) = \Theta([u])$, where $[u]$ is any equivalence class of $S/_{=^{\mathcal{L}}}$. Thus $\Delta \ (=^{\mathcal{L}})^{\dagger} \ \Theta$.

Summary

- A simple proof of the coincidence of bisimilarity with simulation equivalence for reactive systems
- A modal characterisation with a neat completeness proof.