Positive and monotone fragments of FO and LTL

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First-Order Logic (FO)

Signature: Predicate symbols $(P_1, \ldots, P_n)$ with arities $k_1, \ldots, k_n$.

Syntax of FO:

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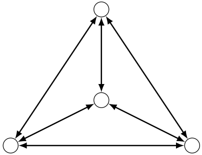
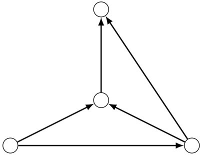
Example: For directed graphs, signature = one binary predicate E.

Graph class	Cliques	No node points to everyone
Formula	$\varphi = \forall x.\forall y.E(x, y)$	$\psi = \neg\exists x.\forall y.E(x, y)$
Example graph	 Model of φ	 Model of ψ

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Motivation: Logics with fixed points.
Fixed points can only be applied to monotone φ.
Hard to recognize $\rightarrow$ replace by positive φ, syntactic condition.

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Theorem (Lyndon 1959)

If φ is monotone then φ is equivalent to a positive formula.

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- [K. 2021,2023]
  EF games on words, elementary

FO on words, the usual way

Words on alphabet $A = \{a, b[, \dots]\}$: signature $(\leq, a, b[, \dots])$

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{a}{\bullet} \longrightarrow \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet}$$

- $x \leq y$ means position x is before position y.
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Examples of formulas:

- $\exists x. a(x)$: words containing a. Language $A^* a A^*$.
- $\exists x, y. (x \leq y \wedge a(x) \wedge b(y))$. Language $A^* a A^* b A^*$.
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Theorem

First-order languages form a strict subclass of regular languages.

Example: $(aa)^*$ is not FO-definable.

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FO-definable languages are well-understood.

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A language $L \subseteq A^*$ is FO-definable iff it is definable by:
Star-free expression $\Leftrightarrow$ LTL $\Leftrightarrow$ counter-free automaton $\Leftrightarrow$ ...

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Intuition: FO languages are "Aperiodic": cannot count modulo
L aperiodic: There is $n \in \mathbb{N}$ such that $\forall u, v, w \in A^*$:

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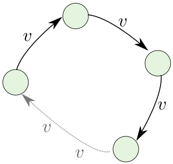

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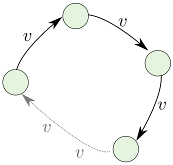

Corollary: FO-definability is decidable for regular languages.

FO on words, the "unconstrained" way

For now, a word is a structure $(X, \leq, a, b, \ldots)$ where

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  $\rightarrow$ Negation necessary for full FO.

The FO^+ logic: positive formulas

FO^+ Logic: a ranges over Σ, no $\neg$

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Question [Colcombet]: FO & monotone $\overset{?}{\Rightarrow}$ FO^+

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Theorem [K. 2021,2023]

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Alphabet $A = \{\emptyset, a, b, c, \binom{a}{b}, \binom{b}{c}, \binom{c}{a}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\}$. Let $a^\uparrow = \{a, \binom{a}{b}, \binom{c}{a}\}$.

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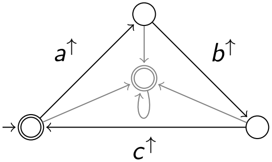

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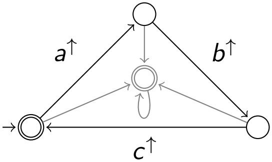

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To prove L is not FO^+-definable: Ehrenfeucht-Fraïssé games.

Can we decide membership?

Theorem
Given L regular on an ordered alphabet, it is decidable whether

- *L is monotone (e.g. automata inclusion)*
- *L is FO-definable [Schützenberger, McNaughton, Papert]*

Can we decide whether L is FO^+-definable ?

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Reduction from *Turing Machine Mortality*:

A deterministic TM M is *mortal* if there a uniform bound n on the runs of M from **any** configuration.

Undecidable [Hooper 1966].

Corollaries: lifting the counter-example

Monotone-$\mathrm{FO} \neq \mathrm{FO}^+$, and FO^+ membership undecidable in the following settings:

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- Words indexed by linear order, finiteness predicate
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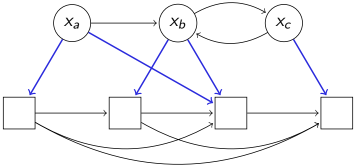
- Words indexed by linear order, finiteness predicate New
- Cost functions on finite words, boundedness predicate contradicts [K. 2011, 2014]

From finite words to finite graphs

Encode words into (directed) graphs, here $ab\binom{a}{b}c$:



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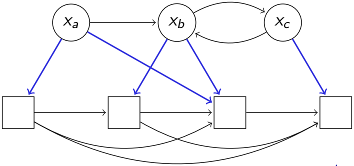

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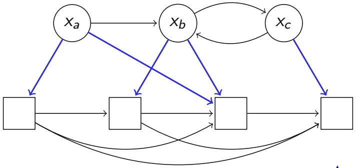

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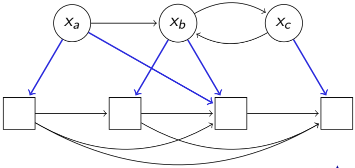

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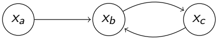
  - $\exists$ 

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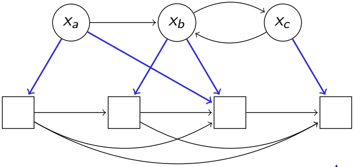

$\rightarrow$ formula ψ_L for graphs encoding words of $L = (a^{\uparrow}b^{\uparrow}c^{\uparrow})^{*} \cup (A^{*} \begin{pmatrix} a \\ b \\ c \end{pmatrix} A^{*})$.

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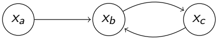
  - $\exists$ 

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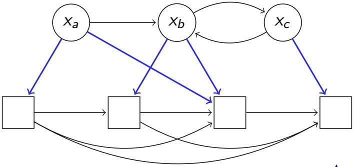

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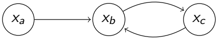
  - $\exists$ 

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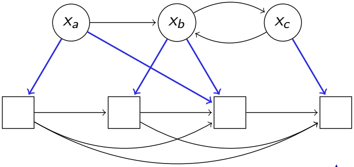

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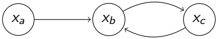
  - $\exists$ 

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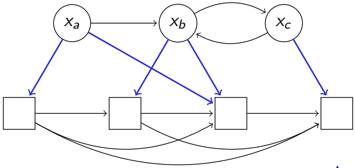

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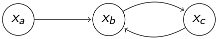
  - $\exists$ 
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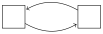
  - ,...

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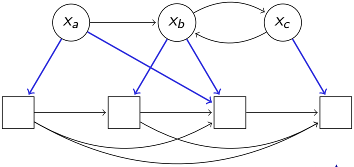

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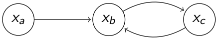
  - $\exists$ 

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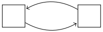
  -  ,...

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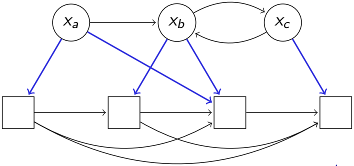

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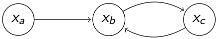
  - $\exists$ 

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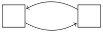
  - 
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Left as exercise: Same with undirected graphs.

Back to words: Link with LTL

LTL syntax:

$$\varphi, \psi ::= \bot \mid \top \mid a \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \mathrm{X}\varphi \mid \varphi \mathrm{U} \psi \mid \varphi \mathrm{R} \psi \mid \neg \varphi.$$

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- $\mathrm{FO} = \mathrm{LTL} = \mathrm{FO}_3$ *[Kamp 1968]*
- $\mathrm{FO}_2[S, <] = \mathrm{UTL}$ *[Etessami, Vardi, Wilke 1997]*
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Theorem (K.,Moreau)

There is no counter-example language definable in $\mathrm{FO}_2[<]$*.*
I.e. $\mathrm{FO}_2[<] \cap$ *Monotone* $\subset \mathrm{FO}^+$*.*

Further work

Open problems::

- $\mathrm{FO}_2 \cap \text{Monotone} \stackrel{?}{=} \mathrm{FO}_2^+$
- For which fragments $F \subset \mathrm{FO}$: $\quad F \cap \text{Monotone} = F^+$
- Other kind of counterexamples ?

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Thanks for your attention !