Positive first-order logic on words and graphs

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First-Order Logic (FO)

Signature: Predicate symbols $(P_1, \ldots, P_n)$ with arities $k_1, \ldots, k_n$.

Syntax of FO:

$$\varphi, \psi := P_i(x_1, \ldots, x_{k_i}) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

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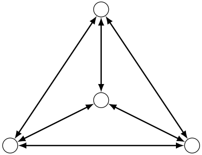
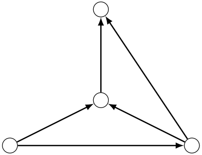
Example: Directed graphs: one binary predicate E.

Graph class	Cliques	No node points to everyone
Formula	$\varphi = \forall x.\forall y.E(x,y)$	$\psi = \neg\exists x.\forall y.E(x,y)$
Example graph	 Model of φ	 Model of ψ

Positive versus Monotone

Goal: Understand the role of negation in FO.

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Motivation: Logics with fixed points.
Fixed points need monotone φ.
$\rightarrow$ positive φ, syntactic condition.

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Theorem (Lyndon 1959)

If φ is monotone then φ is equivalent to a positive formula.

On graph classes: FO-definable+monotone $\Rightarrow$ FO-definable without $\neg$.

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- [This work]
  EF games on words, elementary

Our results

Finite Model Theory:

Lyndon's theorem fails on

- Finite words
- Finite graphs
- Finite structures (elementary proof), several versions:
  - one monotone predicate
  - some monotone predicates
  - all monotone predicates = closure under surjective morphisms.

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Regular Language Theory:

Monotone FO languages	$\neq$	**Positive FO languages**
Algebraic characterization		Logical characterization
Decidable membership		Undecidable membership

FO on words, the usual way

Words on alphabet $A = \{a, b[, \dots]\}$: signature $(\leq, a, b[, \dots])$

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{a}{\bullet} \longrightarrow \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet}$$

- $x \leq y$: position x before position y.
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Examples of formulas:

- $\exists x.a(x)$: Language A^*aA^*.
- $\exists x, y.(x \leq y \wedge a(x) \wedge b(y))$. Language $A^*aA^*bA^*$.

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Theorem

First-order languages form a strict subclass of regular languages.

Example: $(aa)^*$ is not FO-definable. (Proof later)

Background: FO-definable languages

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A language $L \subseteq A^*$ is FO-definable iff it is definable by:
Star-free expression $\Leftrightarrow$ LTL $\Leftrightarrow$ counter-free automaton $\Leftrightarrow$ ...

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Intuition: FO languages are "Aperiodic": cannot count modulo
L aperiodic: There is $n \in \mathbb{N}$ such that $\forall u, v, w \in A^*$:

$$uv^n w \in L \Leftrightarrow uv^{n+1} w \in L.$$

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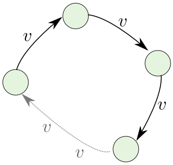

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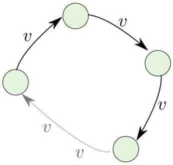

Corollary: FO-definability is decidable for regular languages.

FO on words, the “unconstrained” way

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We will note $\Sigma = \{a, b, \dots\}$, and $A = \mathcal{P}(\Sigma)$ the alphabet.

- Useful e.g. in verification (LTL,...):
  independent signals can be true or false simultaneously.

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  $\rightarrow$ Negation necessary for full FO.

The FO^+ logic: positive formulas

FO^+ Logic: a ranges over Σ, no $\neg$

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Question [Colcombet]: FO & monotone $\overset{?}{\Rightarrow}$ FO^+

A counter-example language

Our first result

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Language $L = (a^\uparrow b^\uparrow c^\uparrow)^* \bigcup A^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} A^*$.

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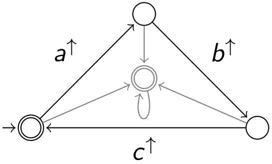
Lemma: L is FO-definable.

Proof: 

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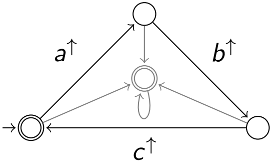
Proof: 

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To prove L is not FO^+-definable: Ehrenfeucht-Fraïssé games.

Ehrenfeucht-Fraïssé games for FO

Definition (EF games)

Played on two words u, v. At each round i:

- Spoiler places token i in u or v.
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Theorem (Ehrenfeucht,Fraïssé, 1950-1961)

L not FO-definable $\Leftrightarrow$ For all n, there are $u \in L$, $v \notin L$ s.t. $u \equiv_n v$.

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Example

Proving $(aa)^*$ is not FO-definable:

$$\begin{array}{lll} u = a^{2k} & \in (aa)^* : & a\ a\ a\ a\ a\ a\ a\ a\ a\ a \\ v = a^{2k-1} & \notin (aa)^* : & a\ a\ a\ a\ a\ a\ a\ a\ a \end{array}$$

Proving FO^+-undefinability

Definition (EF^+ games)

Previous rule: a in $u \Leftrightarrow a$ in v.

We write $u \preceq_n v$ if Duplicator can survive n rounds.

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Application: Proving L is not FO^+-definable

$$\begin{array}{llllllllll} u \in L: & a & b & c & a & b & c & a & b & c \\ v \notin L: & \binom{a}{b} & \binom{b}{c} & \binom{c}{a} & \binom{a}{b} & \binom{b}{c} & \binom{c}{a} & \binom{a}{b} & \binom{b}{c} \end{array}$$

From finite words to finite structures.

Goal: Lift L to finite structures.

For now: signature $(\leq, a, b, c)$ assuming $\leq$ is a total order.

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One monotone predicate

Alphabet encoded by one binary predicate A.

$$a(x) \equiv A(0, x) \qquad b(x) \equiv A(1, x) \qquad c(x) \equiv A(2, x)$$

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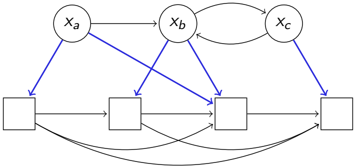

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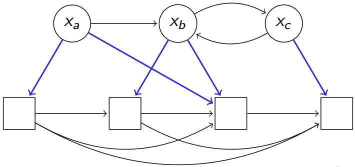

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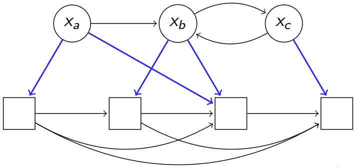

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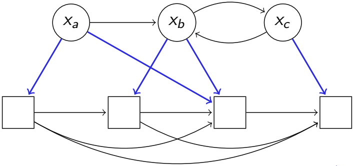

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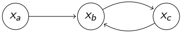
  - 

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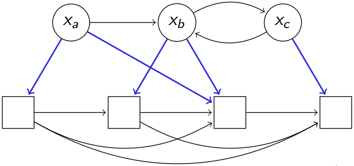

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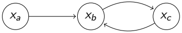
  - 

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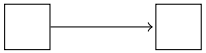
  - 
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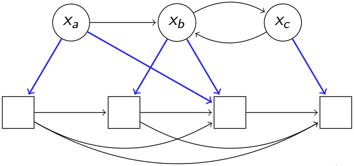

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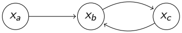
    - 

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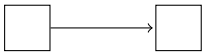
    - 
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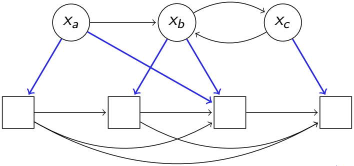

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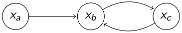
  - 
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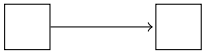
  - ,...
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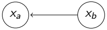
  - 
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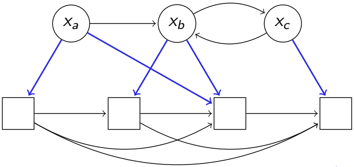

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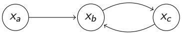
  - 

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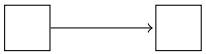
  - 
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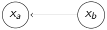
  - 

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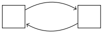
  - 
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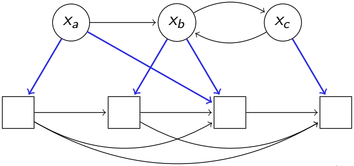

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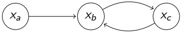
  - 

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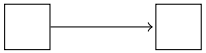
  - 
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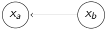
  - 

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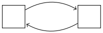
  - 
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Final Formula: $\exists x_a, x_b, x_c.(\psi^- \wedge (\psi_L \vee \psi^+))$

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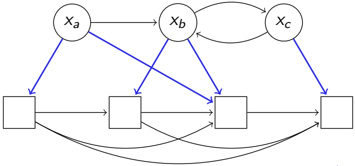

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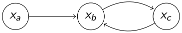
  - 

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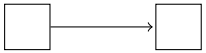
  - ,...
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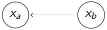
  - 

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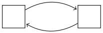
  - ,...

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Left as exercise: Same with undirected graphs.

Back to regular languages

Theorem
Given L regular on an ordered alphabet, it is decidable whether

- *L is monotone (e.g. automata inclusion)*
- *L is FO-definable [Schützenberger, McNaughton, Papert]*

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Reduction from *Turing Machine Mortality*:

A deterministic TM M is *mortal* if there a uniform bound n on the runs of M from **any** configuration.

Undecidable [Hooper 1966].

Undecidability proof sketch

Given a TM M, we build a regular language L such that

$$M \text{ mortal } \Leftrightarrow L \text{ is } \mathrm{FO}^+\text{-definable.}$$

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Building L:
Inspired from $(a^\uparrow b^\uparrow c^\uparrow)^*$, but:

- $a, b, c \rightsquigarrow$ Words from languages C_1, C_2, C_3 encoding configs of M.

- All transitions of M follow the cycle:
$$\begin{tikzcd} & C_2 \arrow[dr] & \\ C_1 \arrow[ur] & & C_3 \arrow[ll] \end{tikzcd}$$

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$u \in L \not\Rightarrow u$ encodes a run of M.

The reduction

If M not mortal:
Let $u_1, u_2, \ldots, u_n$ a long run of M, and play Duplicator in :

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Spoiler always wins in $2n$ rounds $\rightarrow L$ is FO^+-definable.

Ongoing work

With Thomas Colcombet:
Exploring the consequences of this in other frameworks:

- regular cost functions,
- logics on linear orders,
- ...

With Quentin Moreau:

- Links with LTL
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