# CR05: Logic, Automata and Decidability
# Final Exam

Documents are allowed, but not internet. Theorems showed in the course can be used freely.
Questions can be answered in any order. Answers must be justified.
In all the document, $\Sigma$ denotes an arbitrary finite alphabet.

**Exercise 1.** Let $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ be a non-deterministic Büchi automaton on infinite binary trees, where $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of Büchi states, and $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation. Recall that $(p, a, q', q'') \in \Delta$ means that when in state $p$ and reading the letter $a$ on the current node $n$, the automaton can move to the left child of $n$ with state $q'$ and to the right child of $n$ with state $q''$. We assume $\mathcal{A}$ is complete, i.e. for all $(p, a) \in Q \times \Sigma$, there exists $(q', q'') \in Q \times Q$ such that $(p, a, q', q'') \in \Delta$.
We define the *emptiness game* $G_{\mathcal{A}} = (V, E)$ between Players 0 and 1 as follows.
The vertices are $V = V_0 \cup V_1 = Q \cup \Delta$, where $V_0 = Q$ belongs to Player 0 and $V_1 = \Delta$ belongs to Player 1.
The transitions $E \subseteq (V_0 \times V_1) \cup (V_1 \times V_0)$ are defined by: for all $\delta = (p, a, q', q'') \in \Delta$ and $q \in \{q', q''\}$, we have $(p, \delta) \in E$ and $(\delta, q) \in E$.
The winning condition of $G_{\mathcal{A}}$ is the Büchi condition of $\mathcal{A}$, i.e. a play $p_0\delta_0p_1\delta_1p_2\delta_2\ldots$ is winning for Player 0 if and only if the set $\{i \in \mathbb{N} \mid p_i \in F\}$ is infinite.

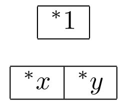
**To do:** Prove that Player 0 wins $G_{\mathcal{A}}$ starting from $q_0$ if and only if $L(\mathcal{A}) \neq \emptyset$.

**Exercise 2.** A language $L \subseteq \Sigma^*$ is *commutative* if for all $u, v \in \Sigma^*$, we have $uv \in L \Leftrightarrow vu \in L$.
A monoid $M$ is *commutative* if for all $x, y \in M$, we have $xy = yx$.

**To do:**

1. Which of the following languages are commutative:
   - $a(a + b)^*a$
   - $\{u \in \{a, b\}^* \mid u \text{ has a prime number of } a\text{'s}\}$
   - $(ab)^*$
2. Let $L$ be a regular language, and $M(L)$ be its syntactic monoid. Prove that if $M(L)$ is commutative then $L$ is commutative.
3. Prove that the converse is not true.
4. Generalize the definition of commutative languages to define the class of strongly commutative languages, such that $L$ is strongly commutative if and only if $M(L)$ is commutative.

**Exercise 3.** Let $M$ be a 3-element aperiodic monoid, given by the following eggbox diagram:




Recall that a star means that the element is idempotent, and that rows represent $\mathcal{R}$-classes.

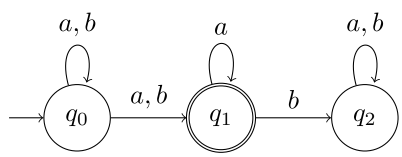
**To do:** Give the list of all languages $L \subseteq \{a, b\}^*$ having $M$ as syntactic monoid.

**Exercise 4.** Let $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ be a Büchi automaton on infinite words. A *choice strategy* for $\mathcal{A}$ is a function $\sigma : \Sigma^* \to Q$ resolving the non-determinism of $\mathcal{A}$ depending on the prefix read so far.
A choice strategy $\sigma$ for $\mathcal{A}$ is *valid* if:

- $\sigma(\varepsilon) = q_0$,
- for all $(u, a) \in \Sigma^* \times \Sigma$, we have $(\sigma(u), a, \sigma(ua)) \in \Delta$,
- for all $w = a_1a_2a_3 \cdots \in L(\mathcal{A})$, the run $\rho_\sigma = \sigma(\varepsilon)\sigma(a_1)\sigma(a_1a_2)\sigma(a_1a_2a_3)\dots$ is accepting.

**To do:**

1. Let $\mathcal{A}$ be the following automaton, with Büchi state doubly circled. Is there a valid choice strategy for $\mathcal{A}$ ?




2. Show that given a Büchi automaton $\mathcal{A}$, it is decidable whether there exists a valid choice strategy for $\mathcal{A}$.

   **Hint**: Define a game $G$ based on $\mathcal{A} \times \mathcal{D}$, where $\mathcal{D}$ is a determinisation of $\mathcal{A}$, such that Player 0 wins $G$ if and only if there exists a valid choice strategy for $\mathcal{A}$.

**Exercise 5.** Let $I = \{a, b\}$ be an input alphabet, $O = \{x, y\}$ be an output alphabet, and $IO = I \times O = \{ax, ay, bx, by\}$ be the specification alphabet.
Let $L \subseteq (IO)^\omega$ be defined by the regular expression $(IO)^*(ax + ay)^\omega$.

**To do:**

1. Give a first-order formula for $L$ on alphabet $IO$.
2. Is the synthesis problem for $L$ satisfiable ? I.e. is there a deterministic letter-to-letter transducer $T : I^\omega \to O^\omega$ such that for all $w \in I^\omega$, the word $(w, T(w))$ of $(IO)^\omega$ (obtained by pairing letters of $w$ and $T(w)$) is in $L$ ?
3. Same two questions with $L'$ defined by $[(ax + by)(ay + bx)]^\omega$.