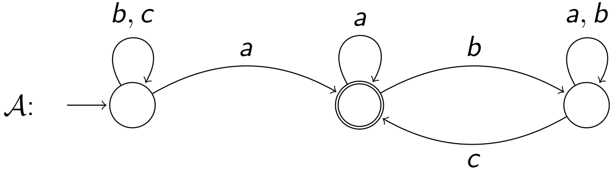
History-deterministic and Explorable Automata

Denis Kuperberg
Marc Bagnol, Udi Boker, Emile Hazard, Olivier Idir, Orna Kupferman, Karoliina Lehtinen, Michał Skrzypczak,. . .

GREYC, Caen, May 14th 2024

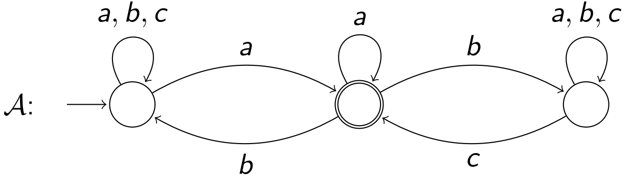
Automata



Deterministic

$$L(\mathcal{A}) = \{\text{accepted words}\}$$

Automata



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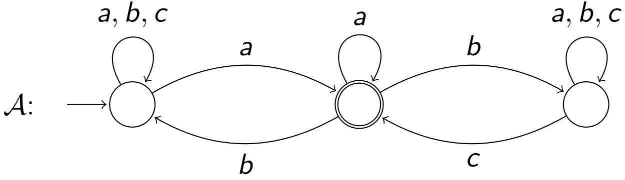

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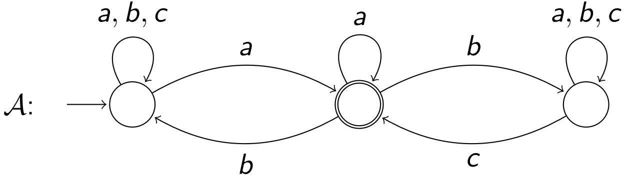

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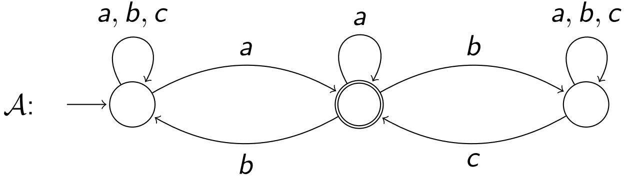

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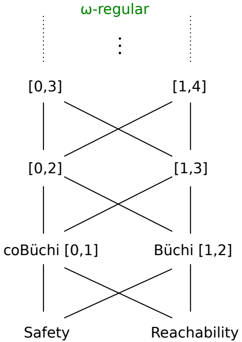
for coBüchi: $L(\mathcal{A}) \subseteq (a + b + c)^* a^\omega$

The Parity hierarchy



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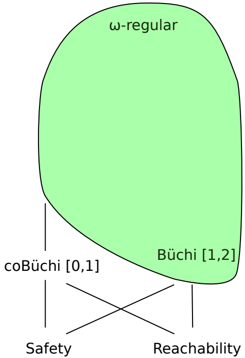

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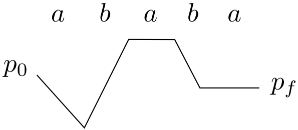
History-Deterministic Automata



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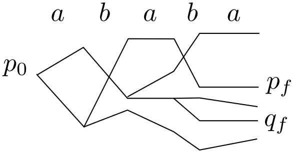

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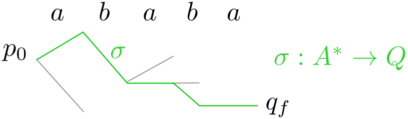

History-Deterministic
(a.k.a. Good-for-Games)

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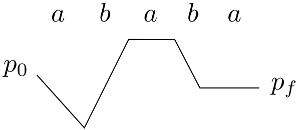

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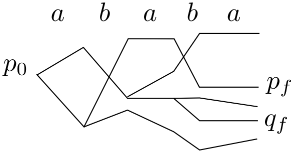

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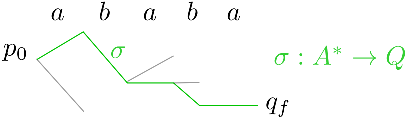

History-Deterministic
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Context

- Introduced in [Henzinger, Piterman 2006] as "Good-for-Games" and in [Colcombet 2009] as "History-determinism".
- Solve Church Synthesis more efficiently
- Intermediate model between Det. and Nondet.

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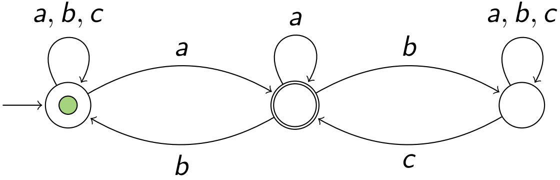
$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters:

Eve: resolves non-deterministic choices for transitions



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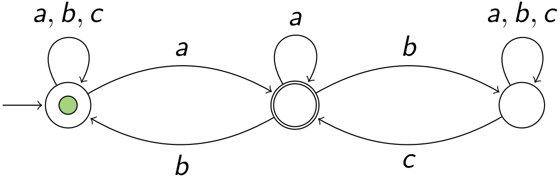

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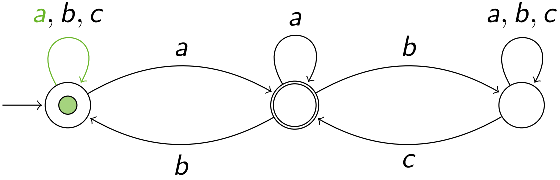

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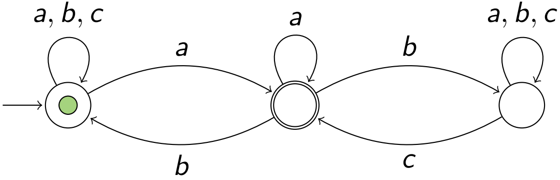

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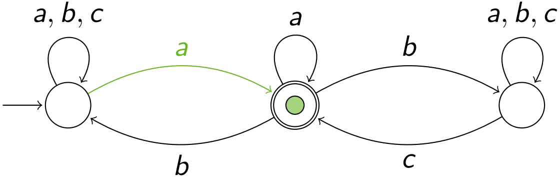

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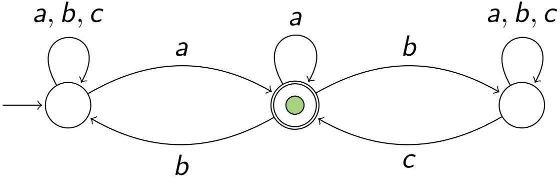

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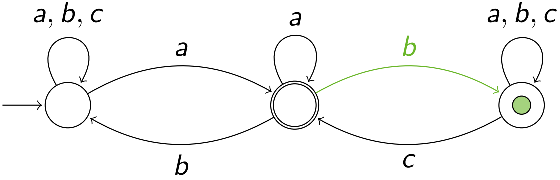

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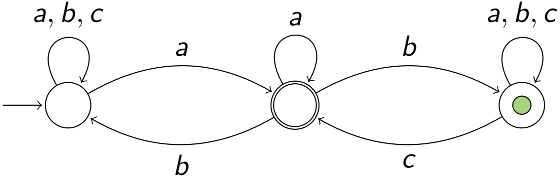

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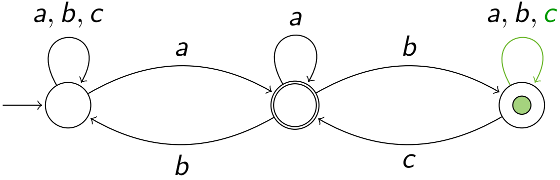

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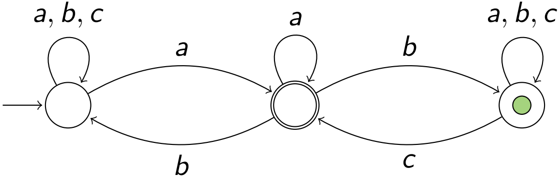

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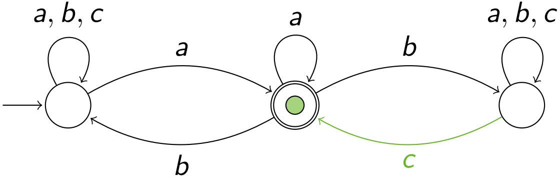

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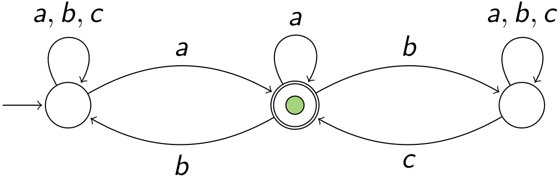

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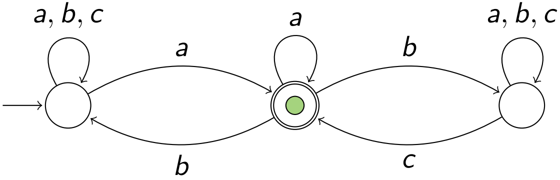

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$\mathcal{A}$ HD $\Leftrightarrow$ Eve wins the Letter game on $\mathcal{A}$

$\Leftrightarrow$ there is a strategy $\sigma_{\mathrm{HD}} : A^* \to Q$ accepting all words of $L(\mathcal{A})$.

First results

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$\mathcal{A}$ DBP if it embeds an equivalent deterministic automaton.

Fact

DBP $\Rightarrow$ HD.

Case where σ_{HD} does not need memory.

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Theorem (Deterministic expressivity)

Any HD automaton can be determinized with exponential blow-up, while preserving its acceptance condition.

Relationship to deterministic automata

Finite words:
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Büchi (aka Parity [1,2]):

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- Determinization in $O(n^2)$ states [K., Skrzypczak 2015], $O(n)$ conjectured.
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coBüchi (aka Parity [0,1]):

- Exponential succinctness of HD vs Det. [K., Skrzypczak 2015]
- PTime minimization [Abu Radi, Kupferman 2020]

Recognizing HD automata

Complexity of the HDness problem:
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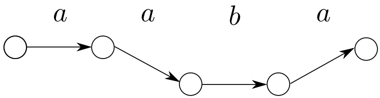
To attack this conjecture and better understand the power of nondeterminism, let us generalize the notion of HD ...

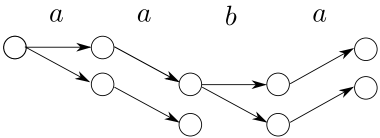
Allowing more runs

Idea: Allow to build several runs, at least one accepting.



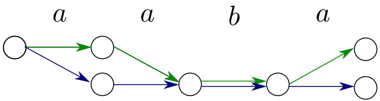
1 state - 1 run

HD



2 states

Width 2
[K., Majumdar 2018]



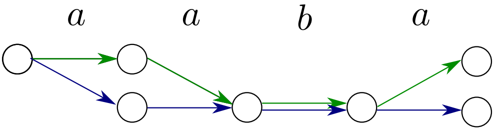
2 runs

2-Explorable
[Hazard, K. 2023]

Explorable Automata

***k*-explorability game**:

Adam plays letters, Eve moves k tokens



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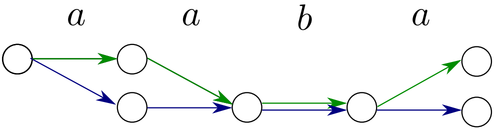

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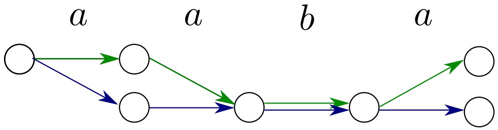

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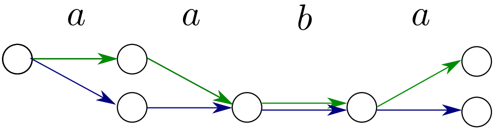

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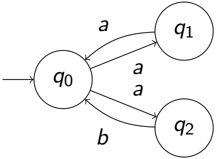

A ?-explorable safety NFA

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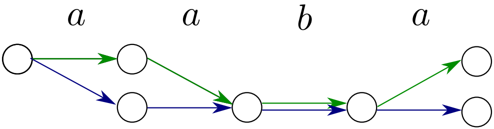

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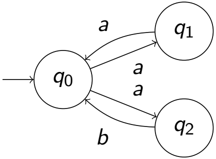

A non-explorable safety NFA

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Theorem [K., Majumdar '18]:
Deciding $|Q|/2$-explorability is EXPTIME-complete.

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Motivating Theorem [Hazard, K. 2023]

The HDness problem is in PTime for explorable automata.

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How many tokens might be needed in explorable automata ?

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like *k*-explorability game on NFA,
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- The PCP is ExpTime-complete
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Our goal: Generalize to Explorability, but

- Game harder to solve: the input word has to be in $L(\mathcal{A})$
- Must deal with acceptance conditions on infinite words.

Results

Theorems [Hazard, K. 2023]

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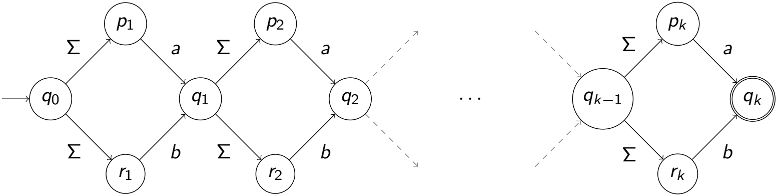
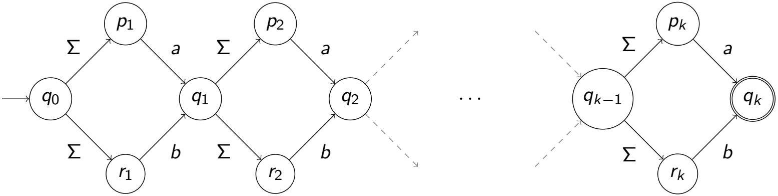


NFA needing exponentially many tokens.

Results

Theorems [Hazard, K. 2023]

Explorability is EXPTIME-complete for NFA, Büchi.
Doubly exponentially many tokens might be needed.



NFA needing exponentially many tokens.

Theorems [Idir, K.]

Explorability is EXPTIME for coBüchi, $[0, 2]$-Parity.

ω-explorability

What happens if we allow a countable infinity of tokens ?

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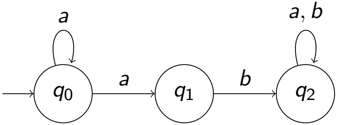
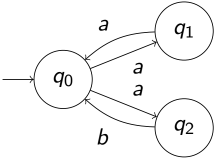


not explorable but
ω-explorable



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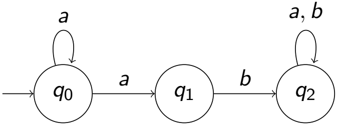

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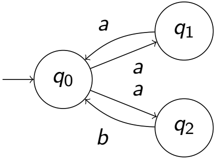

not ω-explorable

Intuition:
Non-ω-explorable: Adam can always kill any run

Results on ω-explorability

Facts:

- any NFA is ω-explorable,
- any automaton $\mathcal{A}$ with $L(\mathcal{A})$ countable is ω-explorable.
- any Reachability automaton is ω-explorable,

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Theorem [Hazard, K. 2023]

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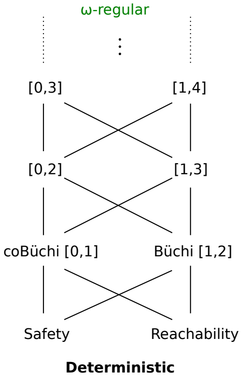
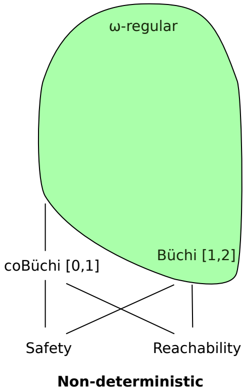
Decidability open for Büchi.

Expressivity of explorable automata





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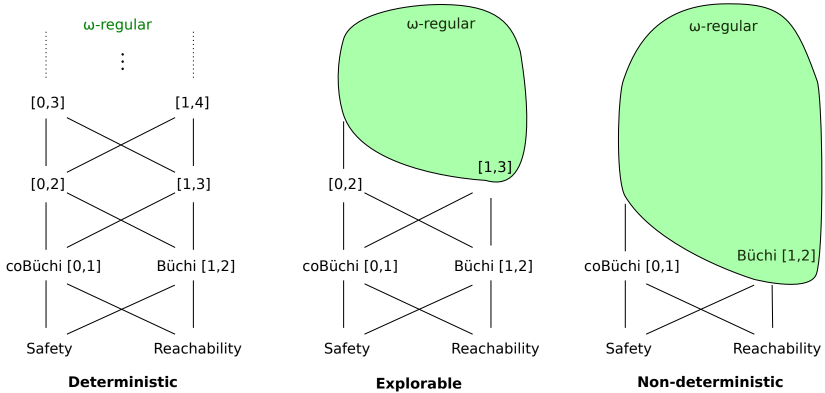

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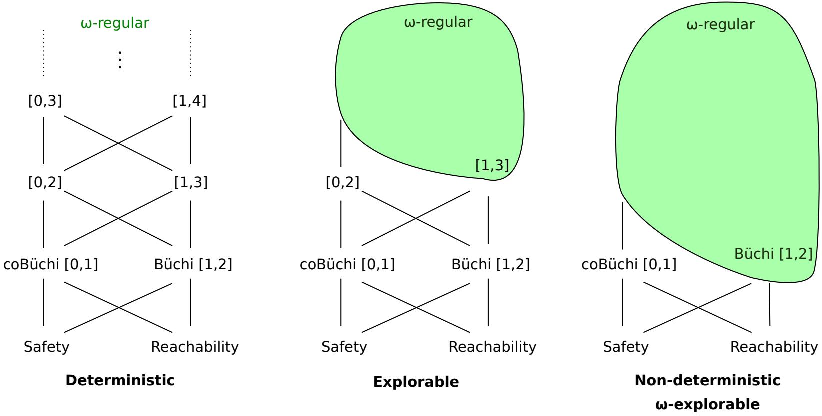

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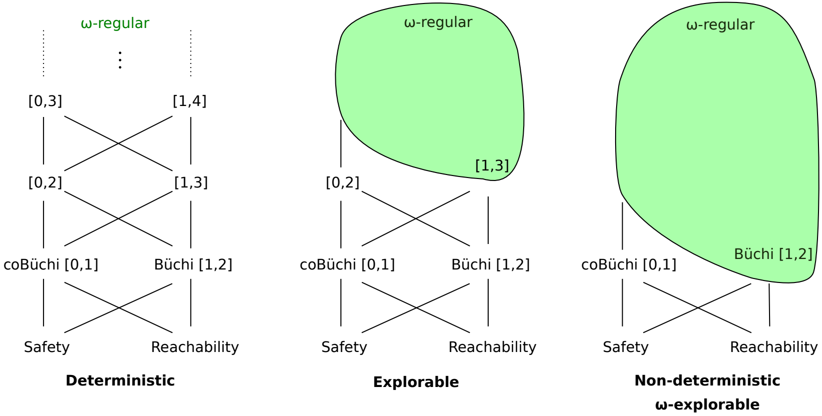

Theorem (Idir, K.)

$[1,3]$-*explorability decidable* $\Leftrightarrow$ *Parity explorability decidable*
Büchi ω-*explorability decidable* $\Leftrightarrow$ *Parity* ω-*explorability decidable*

Future work

- Open decidability: $[1,3]$-expl., Büchi ω-expl.
- Complexity of k-expl. with k in binary?
- Studying HD and expl. models in other frameworks.
- Practical applications, experimental evaluations.
- $\textsc{PTime}$ HDness for parity automata.
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Thanks for your attention!