Positive first-order logic on words and graphs

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Chocola, 10 March 2022

First-Order Logic (FO)

Signature: Predicate symbols $(P_1, \ldots, P_n)$ with arities $k_1, \ldots, k_n$.

Syntax of FO:

$$\varphi, \psi := P_i(x_1, \ldots, x_{k_i}) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

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Structure $(X, R_1, \ldots, R_n)$ is accepted or rejected.

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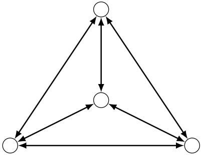
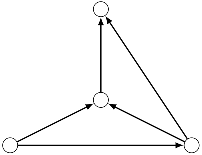
Example: For directed graphs, signature = one binary predicate E.

Graph class	Cliques	No node points to everyone
Formula	$\varphi = \forall x.\forall y.E(x, y)$	$\psi = \neg\exists x.\forall y.E(x, y)$
Example graph	 Model of φ	 Model of ψ

Positive versus Monotone

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Motivation: Logics with fixed points.
Fixed points can only be applied to monotone φ.
Hard to recognize $\rightarrow$ replace by positive φ, syntactic condition.

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Theorem (Lyndon 1959)

If φ is monotone then φ is equivalent to a positive formula.

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- [This work]
  EF games on words, elementary

Our results

Finite Model Theory:

Lyndon's theorem fails on

- Finite words
- Finite graphs
- Finite structures (elementary proof), several versions:
  - one monotone predicate
  - some monotone predicates
  - all monotone predicates = closure under surjective morphisms.

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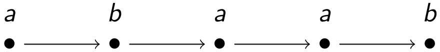
Regular Language Theory:

Monotone FO languages	$\neq$	Positive FO languages
Algebraic characterization		Logical characterization
Decidable membership		Undecidable membership

FO on words, the usual way

Words on alphabet $A = \{a, b[, \dots]\}$: signature $(\leq, a, b[, \dots])$



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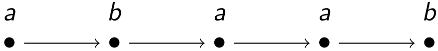

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Examples of formulas:

- $\exists x.a(x)$: words containing a. Language A^*aA^*.
- $\exists x, y.(x \leq y \wedge a(x) \wedge b(y))$. Language $A^*aA^*bA^*$.

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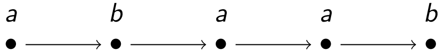

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- $\exists x, y. (x \leq y \wedge a(x) \wedge b(y))$. Language $A^* a A^* b A^*$.

Theorem

First-order languages form a strict subclass of regular languages.

Example: $(aa)^*$ is not FO-definable. (Proof later)

Background: FO-definable languages

FO-definable languages are well-understood.

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Theorem (Schützenberger, McNaughton, Papert)

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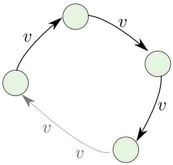

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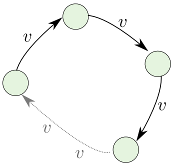

Corollary: FO-definability is decidable for regular languages.

FO on words, the “unconstrained” way

For now, a word is a structure $(X, \leq, a, b, \dots)$ where

- $\leq$ is a total order
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$\rightarrow$ Words on alphabet $\mathcal{P}(\{a, b, \ldots\})$:

$$\overset{\emptyset}{\bullet} \longrightarrow \overset{\{b\}}{\bullet} \longrightarrow \overset{\{a,b\}}{\bullet} \longrightarrow \overset{\emptyset}{\bullet} \longrightarrow \overset{\{b\}}{\bullet}$$

We will note $\Sigma = \{a, b, \ldots\}$, and $A = \mathcal{P}(\Sigma)$ the alphabet.

- Useful e.g. in verification (LTL,...):
  independent signals can be true or false simultaneously.

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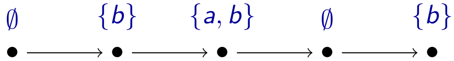

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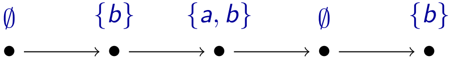

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  $\rightarrow$ Negation necessary for full FO.

The FO^+ logic: positive formulas

FO^+ Logic: a ranges over Σ, no $\neg$

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Question [Colcombet]: FO & monotone $\overset{?}{\Rightarrow}$ FO^+

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Our first result

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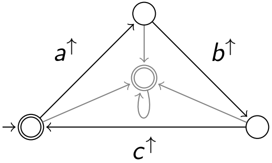
Lemma: L is FO-definable.

Proof: 

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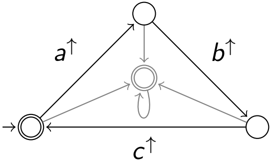

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To prove L is not FO^+-definable: Ehrenfeucht-Fraïssé games.

Ehrenfeucht-Fraïssé games for FO

Definition (EF games)

Played on two words u, v. At each round i:

- Spoiler places token i in u or v.
- Duplicator must answer token i in the other word such that
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Theorem (Ehrenfeucht,Fraïssé, 1950-1961)

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Example

Proving $(aa)^*$ is not FO-definable:

$$u = a^{2k} \quad \in (aa)^* : \quad a\ a\ a\ a\ a\ a\ a\ a\ a\ a$$
$$v = a^{2k-1} \quad \notin (aa)^* : \quad a\ a\ a\ a\ a\ a\ a\ a\ a$$

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New rule:
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Application: Proving L is not FO^+-definable

$$u \in L: \quad a \quad b \quad c \quad a \quad b \quad c \quad a \quad b \quad c$$
$$v \notin L: \quad \binom{a}{b} \binom{b}{c} \binom{c}{a} \binom{a}{b} \binom{b}{c} \binom{c}{a} \binom{a}{b} \binom{b}{c}$$

From finite words to finite structures.

Goal: Lift L to finite structures.

For now: signature $(\leq, a, b, c)$ assuming $\leq$ is a total order.

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$a, b, c, \leq, \not\leq$ are monotone.

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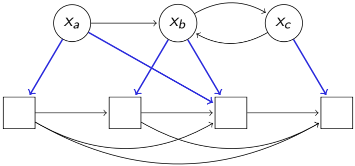
Encode words into (directed) graphs, here $ab\binom{a}{b}c$:



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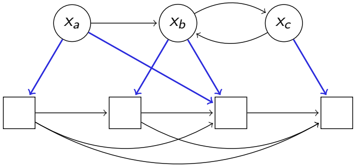

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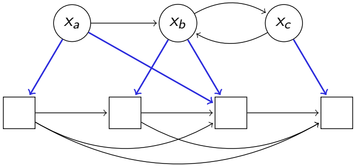

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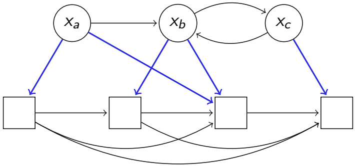

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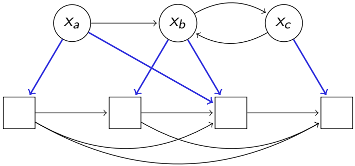

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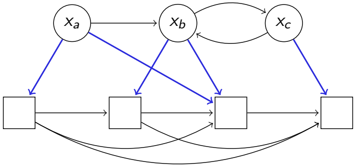

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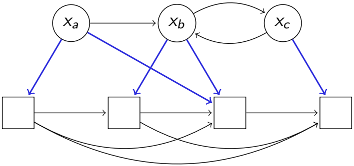

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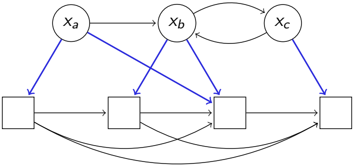

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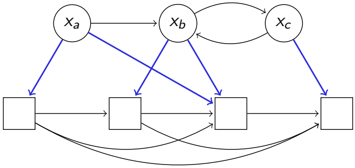

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Final Formula: $\exists x_a, x_b, x_c.(\psi^- \wedge (\psi_L \vee \psi^+))$

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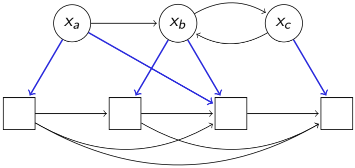

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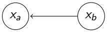
  - 

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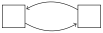
  -  ,...

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Left as exercise: Same with undirected graphs.

Back to regular languages

Theorem
Given L regular on an ordered alphabet, it is decidable whether

- *L is monotone (e.g. automata inclusion)*
- *L is FO-definable [Schützenberger, McNaughton, Papert]*

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Reduction from *Turing Machine Mortality*:

A deterministic TM M is *mortal* if there a uniform bound n on the runs of M from **any** configuration.

Undecidable [Hooper 1966].

Undecidability proof sketch

Given a TM M, we build a regular language L such that

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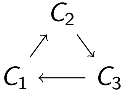
- All transitions of M follow the cycle:



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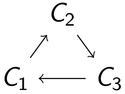

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⚠ $u \in L \not\Rightarrow u$ encodes a run of M.

The reduction

If M not mortal:
Let $u_1, u_2, \ldots, u_n$ a long run of M, and play Duplicator in :

$$\begin{array}{lcccccc} u \in L: & u_1 & u_2 & u_3 & \ldots & u_{n-1} & u_n \\ v \notin L: & \binom{u_1}{u_2} & \binom{u_2}{u_3} & \binom{u_3}{u_4} & \ldots & \binom{u_{n-1}}{u_n} & \end{array}$$

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Play Spoiler in the abstracted game (here $n = 5$):

$$\begin{array}{lllllllllll} u: & 2 & 3 & 2 & 4 & 3 & 5 & 4 & 3 & 4 & 4 \\ v: & \binom{2}{1} & \binom{3}{2} & \binom{2}{1} & \binom{4}{3} & \binom{3}{2} & \binom{5}{4} & \binom{4}{3} & \binom{5}{4} & \binom{5}{4} \end{array}$$

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Spoiler always wins in $2n$ rounds $\rightarrow$ L is FO^+-definable.

Ongoing work

With Thomas Colcombet:
Exploring the consequences of this in other frameworks:

- regular cost functions,
- logics on linear orders,
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Thanks for your attention !