Positive first-order logic on words

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This work was presented at LICS 2021
Long version to appear in LMCS
The $\text{FO}^+$ logic, words as structures

$\text{FO}^+$ Logic: $a$ ranges over $\Sigma$, no $\neg$

$\varphi, \psi := a(x) \mid x \leq y \mid x < y \mid \varphi \lor \psi \mid \varphi \land \psi \mid \exists x. \varphi \mid \forall x. \varphi$
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Word on alphabet $A = 2^\Sigma$:

$$\begin{array}{ccccccc}
\emptyset & \{b\} & \{a, b\} & \emptyset & \{b\} \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet
\end{array}$$
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Word on alphabet $A = 2^\Sigma$:

$$\emptyset \rightarrow \{ b \} \rightarrow \{ a, b \} \rightarrow \emptyset \rightarrow \{ b \}$$

Example: On $\Sigma = \{ a, b \}$:

$$\exists x, y. (x \leq y) \land a(x) \land b(y) \leadsto A^* \{ a \} A^* \{ b \} A^* \cup A^* \{ a, b \} A^*$$
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Word on alphabet $A = 2^\Sigma$:  

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<tr>
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<th>$\emptyset$</th>
<th>${b}$</th>
<th>${a, b}$</th>
<th>$\emptyset$</th>
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$$\exists x, y. (x \leq y) \land a(x) \land b(y) \leadsto A^*\{a\}A^*\{b\}A^* \cup A^*\{a, b\}A^*$$

Remark: $\emptyset^*$ undefinable in $\text{FO}^+$ (cannot say $\neg a$).
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- $\{b\}$
- $\{a, b\}$
- $\emptyset$
- $\{b\}$

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Remark: $\emptyset^*$ undefinable in $\text{FO}^+$ (cannot say "$\neg a$").

More generally: $\text{FO}^+$ can only define monotone languages:

$u\alpha v \in L$ and $\alpha \subseteq \beta \Rightarrow u\beta v \in L$
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More generally: $\text{FO}^+$ can only define monotone languages:

$$u\alpha v \in L \text{ and } \alpha \subseteq \beta \Rightarrow u\beta v \in L$$

Question [Colcombet]: $\text{FO}$ & monotone $\Rightarrow \text{FO}^+$
A counter-example language

Our first result

There is $L$ monotone, FO-definable but not $\text{FO}^+$-definable.
A counter-example language

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There is $L$ monotone, FO-definable but not $\text{FO}^+$-definable.

Alphabet $A = \{\emptyset, a, b, c, (a)_b, (b)_c, (c)_a, (a)_b (b)_c\}$. Let $a^\uparrow = \{a, (a)_b, (c)_a\}$. 
A counter-example language

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Language $L = (a^\uparrow b^\uparrow c^\uparrow)^* \cup A^* (a)_b (b)_c A^*$. 
A counter-example language

Our first result

There is $L$ monotone, FO-definable but not FO$^+$-definable.

Alphabet $A = \{\emptyset, a, b, c, \left(\begin{array}{c} a \\ b \end{array}\right), \left(\begin{array}{c} b \\ c \end{array}\right), \left(\begin{array}{c} a \\ c \end{array}\right)\}$. Let $a^{\uparrow} = \{a, \left(\begin{array}{c} a \\ b \end{array}\right), \left(\begin{array}{c} c \\ a \end{array}\right)\}$.

Language $L = (a^{\uparrow}b^{\uparrow}c^{\uparrow})^* \cup A^* \left(\begin{array}{c} a \\ b \\ c \end{array}\right) A^*$.

Lemma: $L$ is FO-definable.

Proof: $a^{\uparrow}$, $b^{\uparrow}$, and $c^{\uparrow}$ is counter-free. (no cycle labelled $u \geq 2$)
A counter-example language

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There is $L$ monotone, FO-definable but not FO$^+$-definable.

Alphabet $A = \{\emptyset, a, b, c, \left(\frac{a}{b}\right), \left(\frac{b}{c}\right), \left(\frac{c}{a}\right), \left(\frac{a}{b} c\right), \left(b \frac{c}{a}\right), \left(c \frac{a}{b}\right), \left(a \frac{b}{c}\right)\}$. Let $a^\uparrow = \{a, \left(\frac{a}{b}\right), \left(\frac{c}{a}\right)\}$.

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Lemma: $L$ is FO-definable.

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Syntactic monoid of \( L \)

\[
\begin{array}{ccc}
(\ a \ b) & (a \ b) (b) & (b) (c) (a) \\
(b) (c) (a) (b) & (b) (c) & (b) (c) \\
(c) (a) (b) & (c) (a) (b) & (c) (a) \\
\end{array}
\]

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It remains to prove that \( L \) is not \( \text{FO}^+ \)-definable.
Syntactic monoid of $L$

$$
\begin{array}{ccc}
(a) & (a)(b) & (a)(b)(c) \\
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\end{array}
\begin{array}{ccc}
(b)(c)(a)(b) & (b)(c) & (b)(c)(a)
\end{array}
\begin{array}{ccc}
(c)(a)(b) & (c)(a)(b)(c) & (c)(a)
\end{array}
\begin{array}{ccc}
(a) & (a)(b) & (a)(b)(c)
\end{array}
$$

$$
\begin{array}{ccc}
a & ab & abc \\
bca & b & bc \\
ca & cab & c \\
\emptyset & T & \top
\end{array}
$$

It remains to prove that $L$ is not $\mathsf{FO}^+$-definable.
Ehrenfeucht-Fraïssé games for FO

Definition (EF games)

Played on two words $u$, $v$. At each round $i$:

- **Spoiler** places token $i$ in $u$ or $v$.
- **Duplicator** must answer token $i$ in the other word such that
  - the letter on token $i$ is the same in $u$ and $v$.
  - the tokens are in the same order in $u$ and $v$. 

Let us note $u \equiv_n v$ if Duplicator can survive $n$ rounds on $u$, $v$. 

Theorem (Ehrenfeucht, Fraïssé, 1950-1961)

$L$ not FO-definable $\iff$ For all $n$, there are $u \in L$, $v \not\in L$ s.t. $u \equiv_n v$.

Example

Proving $(aa)^*$ is not FO-definable:

$u = a^{2k} \in (aa)^*$:

$a a a a a a a a a a$

$v = a^{2k-1} \not\in (aa)^*$:

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Proving \( \text{FO}^+ \)-undefinability

**Definition (\( \text{EF}^+ \) games)**

New rule:
Letters in \( u \) just have to be **included** in corresponding ones in \( v \).

We write \( u \preceq_n v \) if Duplicator can survive \( n \) rounds.

Theorem (Correctness of \( \text{EF}^+ \) games)

\[ L \not\text{FO}^+\text{-definable} \iff \forall n, \text{there are } u \in L, v \not\in L \text{ s.t. } u \preceq_n v. \]

Application: Proving \( L \) is not \( \text{FO}^+\)-definable

\[ u \in L: a \ b \ c \ a \ b \ c \ a \ b \ c \]

\[ v \not\in L: (a \ b)(b \ c)(c \ a)(a \ b)(b \ c)(c \ a)(a \ b)(b \ c) \]
Proving $\text{FO}^+\text{-undefinability}$

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New rule:
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**Theorem (Correctness of EF$^+$ games)**

$L$ not $\text{FO}^+$-definable $\iff \forall n$, there are $u \in L$, $v \notin L$ s.t. $u \preceq^*_n v$.

[Stolboushkin 1995+this work]
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**Application: Proving \( L \) is not \( \text{FO}^+ \)-definable**

\[
\begin{align*}
  u \in L : & \quad a \ b \ c \ a \ b \ c \ a \ b \ c \\
  v \notin L : & \quad (a)_b \ (b)_c \ (c)_a \ (a)_b \ (b)_c \ (c)_a \ (a)_b \ (b)_c 
\end{align*}
\]
Background: Lyndon’s theorem

First-order logic on arbitrary structures, signature \((P_1, \ldots, P_k)\).

**Theorem (Lyndon 1959)**

Let \(\varphi \in \text{FO}\), stable under making predicates true on more tuples. Then \(\varphi\) is equivalent to a negation-free formula.

**Example:** If a language of graphs is FO-definable and closed under adding edges, then it is FO-definable without \(\neg\).
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Lyndon’s theorem fails on finite structures:

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  lattices, probabilities, number theory, topology, very hard

- [Stolboushkin 1995]
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- [This work]
  EF games on words, elementary thanks to $L$
Can we decide $\text{FO}^+$-definability?

**Theorem**

*Given* $L$ *regular on an ordered alphabet, we can decide*

- whether $L$ is monotone (e.g. automata inclusion)
- whether $L$ is FO-definable [Schützenberger, McNaughton, Papert]

Can we decide whether $L$ is $\text{FO}^+$-definable?
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$\text{FO}^+$-definability is undecidable for regular languages.
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**Our second result**

$\text{FO}^+$-definability is undecidable for regular languages.

Reduction from *Turing Machine Mortality*:
A deterministic TM $M$ is *mortal* if there a uniform bound $n$ on the runs of $M$ from any configuration.

Undecidable [Hooper 1966].
Undecidability proof sketch

Given a TM $M$, we build a regular language $L$ such that

$$M \text{ mortal} \iff L \text{ is } \text{FO}^{+}-\text{definable}.$$
Undecidability proof sketch

Given a TM $M$, we build a regular language $L$ such that

$$M \text{ mortal } \iff L \text{ is } \text{FO}^+\text{-definable.}$$

**Building $L$:**
Inspired from $(a \uparrow b \uparrow c \uparrow)^*$, but:

- $a, b, c \leadsto$ Words from $C_1, C_2, C_3$ encoding configs of $M$.

- All transitions of $M$ follow the cycle: $C_1 \leftarrow C_2 \leftarrow C_3$

- $(a \ b), (b \ c), (c \ a) \leadsto (u_1 \ u_2)$, exists iff $u_1 \xrightarrow{M} u_2$. 
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Given a TM \( M \), we build a regular language \( L \) such that

\[
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- \( a, b, c \rightsquigarrow \) Words from \( C_1, C_2, C_3 \) encoding configs of \( M \).
- All transitions of \( M \) follow the cycle:

\[
\begin{array}{c}
\uparrow \\
C_1 \\
\downarrow \\
C_2 \\
\downarrow \\
C_3 \\
\uparrow \\
C_1
\end{array}
\]

- \((a_b), (b_c), (c_a) \rightsquigarrow (u_1 \underline{u_2})\), exists iff \( u_1 \xrightarrow{M} u_2 \).

We choose

\[
L := (C_1 \uparrow \cdot C_2 \uparrow \cdot C_3 \uparrow)^*
\]
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\[
\begin{array}{c}
C_1 \\
\downarrow \quad \uparrow \\
C_2 \\
\downarrow \quad \leftarrow \quad \rightarrow \\
C_3
\end{array}
\]

- $(a_b, b_c, c_a) \mapsto (u_1^{u_2}),$ exists iff $u_1 \xrightarrow{M} u_2$.

We choose

$$L := (C_1^{↑} \cdot C_2^{↑} \cdot C_3^{↑})^*$$

⚠️ $u \in L \not\Rightarrow u$ encodes a run of $M$. 
The reduction

If \( M \) not mortal:
Let \( u_1, u_2, \ldots, u_n \) a long run of \( M \), and play Duplicator in:

\[
\begin{align*}
&u \in L: \quad u_1 \ u_2 \ u_3 \ \ldots \ u_{n-1} \ u_n \\
&v \not\in L: \quad \binom{u_1}{u_2} \binom{u_2}{u_3} \binom{u_3}{u_4} \ \ldots \ \binom{u_{n-1}}{u_n}
\end{align*}
\]

\( \rightarrow L \) is not \( \text{FO}^+ \)-definable.
The reduction

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\end{align*}
\]

\( \rightarrow L \) is not \( \text{FO}^+ \)-definable.

If \( M \) mortal with bound \( n \):
Abstract \( u_i \) by the length of the run of \( M \) starting in it (at most \( n \)).
The reduction

If $M$ not mortal:
Let $u_1, u_2, \ldots, u_n$ a long run of $M$, and play Duplicator in:

$$\begin{align*}
u \in L &: u_1 \ u_2 \ u_3 \ \ldots \ \ u_{n-1} \ u_n \\
v \notin L &: (u_1 \ u_2) \ (u_2 \ u_3) \ (u_3 \ u_4) \ \ldots \ (u_{n-1} \ u_n)
\end{align*}$$

$\rightarrow L$ is not $\text{FO}^+$-definable.

If $M$ mortal with bound $n$:
Abstract $u_i$ by the length of the run of $M$ starting in it (at most $n$).

Play Spoiler in the abstracted game (here $n = 5$):

$$\begin{align*}
u : & \quad 2 \quad 3 \quad 2 \quad 4 \quad 3 \quad 5 \quad 4 \quad 3 \quad 4 \quad 4 \\
v : & \quad (2 \ 1) \quad (3 \ 2) \quad (2 \ 1) \quad (4 \ 3) \quad (3 \ 2) \quad (5 \ 4) \quad (4 \ 3) \quad (5 \ 4) \quad (5 \ 4)
\end{align*}$$
The reduction

If \( M \) not mortal:
Let \( u_1, u_2, \ldots, u_n \) a long run of \( M \), and play Duplicator in :

\[
\begin{align*}
  u &\in L : \quad u_1 \quad u_2 \quad u_3 \quad \ldots \quad u_{n-1} \quad u_n \\
  v &\notin L : \quad (u_1^{u_2}) \quad (u_2^{u_3}) \quad (u_3^{u_4}) \quad \ldots \quad (u_{n-1}^{u_n})
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Play Spoiler in the abstracted game (here \( n = 5 \)):

\[
\begin{align*}
  u : &\quad 2 \quad 3 \quad 2 \quad 4 \quad 3 \quad 5 \quad 4 \quad 3 \quad 4 \quad 4 \\
  v : &\quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix}
\end{align*}
\]

Spoiler always wins in \( 2n \) rounds \( \rightarrow L \) is FO\(^+\)-definable.
Ongoing work

For the long version:
The counter-example can be encoded into graphs
→ Lyndon’s theorem fails on finite graphs.

With Thomas Colcombet:
Exploring the consequences of this in other frameworks:
▶ regular cost functions,
▶ logics on linear orders,
▶ ...

Slogan:
FO variants without negation will often display this behaviour.
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The counter-example can be encoded into graphs
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Thanks for your attention!