Explorable automata

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Abstract

We define the class of explorable automata on finite or infinite words. This is a generalization of Good-For-Games (GFG) automata, where this time non-deterministic choices can be resolved by building finitely many simultaneous runs instead of just one. We show that recognizing GFG automata among explorable ones is in PTime, thereby giving a strong link between the two notions. We then show that recognizing explorable automata is ExpTime-complete, in the case of finite words or Büchi automata. Additionally, we define the notion of ω-explorable automata on infinite words, where countably many runs can be used to resolve the non-deterministic choices. We show that all reachability automata are ω-explorable, but this is not the case for safety ones. We finally show ExpTime-completeness for ω-explorability of automata on infinite words, covering the safety and co-Büchi acceptance conditions.

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1 Introduction

In several fields of theoretical science, the tension between deterministic and non-deterministic models is a source of fundamental open questions, and has led to important lines of research. The most famous of this kind is the P vs NP question in complexity theory. This paper aims at further investigating the frontier between determinism and non-determinism in automata theory. Although Non-deterministic and Deterministic Finite Automata (NFA and DFA) are known to be equivalent, many subtle questions remain about the cost of determinism, and a deep understanding of non-determinism will be needed to solve them.

One of the approaches investigating non-determinism in automata is the study of Good-For-Games (GFG) automata, introduced in [13]. An automaton is GFG if when reading input letters one by one, its non-determinism can be resolved on-the-fly without any need to guess the future. This constitutes a model that is intermediary between non-determinism and determinism, and can sometimes bring the best of both worlds. Like deterministic automata, GFG automata on infinite words retain good properties such as their soundness with respect to composition with games, making them appropriate for use in Church synthesis algorithms [13]. On the other hand, like non-deterministic automata, they can be exponentially more succinct than deterministic ones [16]. There is a very active line of research trying to understand the various properties of GFG automata, see *e.g.* [21, 22, 4, 6, 17, 11, 23] for latest developments. Notice that GFG automata are also called *history-deterministic*, a terminology introduced originally in the theory of regular cost functions [9]. The name "history-deterministic" corresponds to the above intuition of solving non-determinism on-the-fly, while "good-for-games" refers to sound composition with games. These two notions may actually differ in some quantitative frameworks, but coincide on boolean automata [5].

The goal of this paper is to pursue this line of research by introducing and studying the class of explorable automata on finite and infinite words. The intuition behind explorability is to limit the amount of non-determinism required by the automaton to accept its language,

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in a more permissive way than GFG automata. If $k \in \mathbb{N}$, an automaton is k-explorable if when reading input letters, it suffices to keep track of k runs to build an accepting one, if it exists. An automaton is explorable if it is k-explorable for some $k \in \mathbb{N}$. This can be seen as a variation on the notion of GFG automaton, which corresponds to the case $k = 1$. The present work can be compared to [15], where a notion related to k-explorability (called *width*) is introduced and studied. In [15], the notion of simultaneous runs is different and more permissive, and does not give any meaningful notion of explorability, because n simultaneous runs always suffice for an automaton with n states. However some results of [15] also apply to k-explorability, notably ExpTime-completeness of deciding k-explorability of an NFA if k is part of the input. Surprisingly however, the techniques used in [15] are quite different from the ones we need here. This shows that fixing a bound k for the number of runs leads to very different problems compared to asking for the existence of such a bound.

One of the motivations to introduce the notion of explorability is to tackle one of the important open questions about GFG automata: what is the complexity of deciding whether an automaton is GFG? Recognizing GFG automata is known to be in PTime for Büchi [1] and co-Büchi [16] automata, but even for 3 parity ranks, the only known upper bound is ExpTime via the naive algorithm from [13]. We show how explorable automata can simplify this question: if the input automaton is explorable, then the problem becomes PTime. Therefore, the question of recognizing GFG automata can be shifted to: how hard is it to recognize explorable automata?

We then proceed to study the decidability and complexity of the explorability problem: deciding whether an input automaton on finite or infinite words is explorable. For this, we establish a connection with the population control problem studied in [2]. This problem asks, given an NFA with an arbitrary number of tokens in the initial state, whether a controller can choose input letters, thereby forcing every token to reach a designated state, even if tokens are controlled by an opponent. It is shown in [2] that the population control problem is ExpTime-complete, and we adapt their proof to our setting to show that the explorability problem is ExpTime-complete as well, already for NFAs. We also show that a direct reduction is possible, but at an exponential cost, yielding only a 2-ExpTime algorithm for the NFA explorability problem. In the case of infinite words, we adapt the proof to the Büchi case, thereby showing that the Büchi explorability problem is in ExpTime as well. We also remark that as in [2], the number of tokens needed to witness explorability can go as high as doubly exponential in the size of the automaton.

This ExpTime-completeness result means that we unfortunately cannot directly use the intermediate notion of explorable automata to improve on the complexity of recognizing GFG automata, as could have been the hope. We still believe however that this explorability notion is of interest towards a better understanding of non-determinism in automata theory.

Notice that interestingly, from a model-checking perspective, our approach is dual to [2]: in the population control problem, an NFA is well-behaved when we can “control” it by forcing arbitrarily many runs to simultaneously reach a designated state, via an appropriate choice of input letters. On the contrary, in our approach, the input letters form an adversarial environment, and our NFA is well-behaved when its non-determinism is limited, in the sense that it is enough to spread finitely many runs to explore all possible behaviours.

On infinite words, we push further the notion of explorability, by remarking that for some automata, even following a countable number of runs is not enough. This leads to defining the class of *ω-explorable automata*, as those automata on infinite words where non-determinism can be resolved using countably many runs. We show that ω-explorable automata form a non-trivial class even for the safety acceptance condition (but not for reachability), and give an ExpTime algorithm recognizing ω-explorable automata, encompassing the safety and co-Büchi conditions. We also show ExpTime-hardness of this problem, by adapting the ExpTime-hardness proof of [2] to the setting of ω-explorability.

Summary of the contributions. We show that given an explorable parity automaton of fixed parity index, it is in PTime to decide whether it is GFG. The algorithm used for Büchi in [1] is conjectured to work for any acceptance condition (this is the "G_2 conjecture"), and it is in fact this algorithm that is shown here to work on any explorable parity automaton.

We show that given a NFA or Büchi automaton, it is decidable and ExpTime-complete to check whether it is explorable. Our proof of ExpTime-completeness for NFAs uses techniques developed in [2], where ExpTime-completeness is shown for the NFA population control problem. We generalize this result to ExpTime explorability checking for Büchi automata, requiring further adaptations. We also give a black box reduction using the result from [2]. This is enough to show decidability of the NFA explorability problem, but it yields a 2-ExpTime algorithm. As in [2], the ExpTime algorithm yields a doubly exponential tight upper bound on the number of tokens needed to witness explorability.

On infinite words, we show that any reachability automaton is ω-explorable, but that this is not the case for safety automata. We show that both the safety and co-Büchi ω-explorability problems are ExpTime-complete.

Related Works. Many works aim at quantifying the amount of non-determinism in automata. A survey by Colcombet [10] gives useful references on this question. Let us mention for instance the notion of ambiguity, which quantifies the number of simultaneous accepting runs. Similarly as in [15], we can note that ambiguity is orthogonal to k-explorability. Remark however that our finite/countable/uncountable explorability hierarchy is reminiscent of the finite/polynomial/exponential ambiguity hierarchy (see *e.g.* [24]).

In [14], several ways of quantifying the non-determinism in automata are studied from the point of view of complexity, including notions such as the number of advice bits needed.

Another approach is studied in [20], where a measure of the maximum non-deterministic branching along a run is defined and compared to other existing measures.

Following the GFG approach, a hierarchy of non-determinism and an analysis of this hierarchy via probabilistic models is given in [22].

We define explorability via games with tokens inspired by the approach in [1]. These games with tokens and their interplay with various quantitative acceptance conditions were recently investigated in [6].

2 Explorable automata

2.1 Preliminaries

If $i \leq j$ are integers, we will denote by $[i,j]$ the integer interval $\{i, i+1, \dots, j\}$. If S is a set, its cardinal will be denoted $|S|$, and its powerset $\mathcal{P}(S)$.

We work with a fixed finite alphabet Σ. We will use the following default notation for the components of an automaton $\mathcal{A}$: $Q_{\mathcal{A}}$ for its states of states, $q_0^{\mathcal{A}}$ for its initial state, $F_{\mathcal{A}}$ for its accepting states, $\Delta_{\mathcal{A}}$ for its set of transitions. The subscript might be omitted when clear from context. We might also specify its alphabet by $\Sigma_{\mathcal{A}}$ instead of Σ for cases where different alphabets come into play. If $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $(p,a) \in Q \times \Sigma$, we will note $\Delta(p,a) = \{q \in Q, (p,a,q) \in \Delta\}$. If $X \subseteq Q$, we note $\Delta(X,a) = \bigcup_{p \in X} \Delta(p,a)$.

We will consider non-deterministic automata on finite words (NFAs). A run of such an automaton on a word $a_1 a_2 \dots a_n \in \Sigma^*$ is a sequence of states $q_0 q_1 \dots q_n \in Q^*$ (q_0 being the initial state), such that for all $i \in [0, n-1]$, we have $q_{i+1} \in \Delta(q_i, a_i)$. Such a run is accepting if $q_n \in F$, *i.e.* if the run belongs to Q^*F. As usual, the language of an automaton $\mathcal{A}$, denoted $L(\mathcal{A})$, is the set of words that admit an accepting run.

We will also deal with automata on infinite words, and we recall here some of the standard acceptance conditions for such automata. A run on an infinite word $w \in \Sigma^\omega$ is now an infinite sequence of states, *i.e.* an element of Q^ω, starting in q_0 and following as before transitions of the automaton according to the letters of w. Such a run of Q^ω is accepting in a safety (resp. reachability, Büchi, co-Büchi) automaton if it belongs to F^ω (resp. Q^*FQ^ω, $(Q^*F)^\omega$, Q^*F^ω). States from F will be called Büchi states in Büchi automata, and states from $Q \setminus F$ will be called co-Büchi states in co-Büchi automata.

Finally, we will also mention the parity acceptance condition: it uses a ranking function rk from Q to an interval of integers $[i, j]$. A run is accepting if the minimal rank appearing infinitely often is even (following the convention of [2]).

2.2 Explorability

We start by introducing the *k-explorability game*, which is the central tool allowing us to define the class of explorable automata.

▶ **Definition 1** (k-explorability game)**.** *Consider a non-deterministic automaton $\mathcal{A}$ on finite or infinite words, and an integer k. The k-explorability game on $\mathcal{A}$ is played on the arena Q^k. The two players are called Determiniser and Spoiler, and they play as follows.*

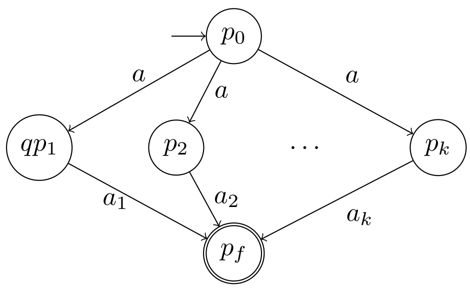
- *The initial position is the k-tuple $S_0 = (q_0, \dots, q_0)$.*
- *At step i from a position $S_{i-1} \in Q^k$, Spoiler chooses a letter $a_i \in \Sigma$, and Determiniser chooses $S_i \in Q^k$ such that for any token $l \in [0, k-1]$, $S_{i-1}(l) \xrightarrow{a} S_i(l)$ is a transition of $\mathcal{A}$ (where $S_i(l)$ stands for the l-th component in S_i).*

The play is won by Determiniser if for any $\beta \leq \omega$ such that the word $(a_i)_{1 \leq i < \beta}$ is in $\mathcal{L}(\mathcal{A})$, there is a token $l < k$ being accepted by $\mathcal{A}$, meaning that the sequence $(S_i(l))_{i<\beta}$ is an accepting run[1]*. Otherwise the winner is Spoiler.*
We will say that $\mathcal{A}$ is k-explorable *if Determiniser wins the k-explorability game.*
We will say that $\mathcal{A}$ is explorable *if it is k-explorable for some $k \in \mathbb{N}$.*

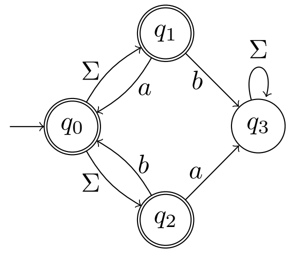
▶ **Example 2.** The NFA $\mathcal{A}_k$ on alphabet $\{a, a_1, \dots, a_k\}$ is k-explorable but not $(k-1)$-explorable. It can easily be adapted to a binary alphabet, by replacing in the automaton $a_1, \dots, a_k$ by distinct words of the same length.

On the other hand, the NFA $\mathcal{C}$ is a non-explorable NFA accepting all words on alphabet $\Sigma = \{a, b\}$. Indeed, Spoiler can win the k-explorability game for any k, by eliminating tokens one by one, choosing at each step the letter b if q_1 is occupied by at least one token, and the letter a otherwise.

[1] This condition $\beta \leq \omega$ is actually accounting separately for the two cases of finite and infinite words, corresponding respectively to $\beta < \omega$ and $\beta = \omega$.

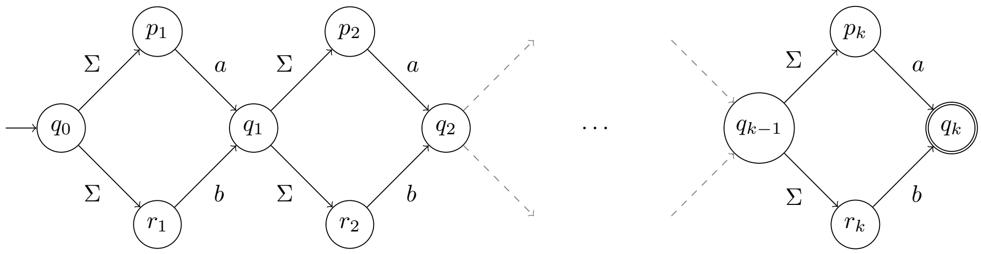


Explorable $\mathcal{A}_k$



Non-explorable $\mathcal{C}$

▶ **Example 3.** The following NFA $\mathcal{B}_k$ with $3k+1$ states on alphabet $\Sigma = \{a, b\}$ is explorable but requires 2^k tokens. Indeed, since when choosing the $2i^{\text{th}}$ letter Spoiler can always pick the state p_i or r_i containing the least amount of tokens to decide whether to play a or b, the best strategy for Determiniser is to split his tokens evenly at each q_i. This means he needs to start with 2^k tokens to end up with at least one token in q_k after a word of Σ^{2k}.



Let us mention a few facts that follow from the definition of explorability:

▶ **Lemma 4.**

- *Any finite language is explorable.*
- *If $\mathcal{A}$ is k-explorable, then it is n-explorable for all $n \geq k$.*
- *If $\mathcal{A}$ is k-explorable and $\mathcal{B}$ is n-explorable, then*
  - *$\mathcal{A} \cup \mathcal{B}$ (with states $Q = \{q_0\} \cup Q_{\mathcal{A}} \cup Q_{\mathcal{B}}$) is $(k+n)$-explorable,*
  - *the union product $\mathcal{A} \times \mathcal{B}$ (with $F = (F_{\mathcal{A}} \times Q_{\mathcal{B}}) \cup (Q_{\mathcal{A}} \times F_{\mathcal{B}})$) is $\max(k, n)$-explorable,*
  - *the intersection product $\mathcal{A} \times \mathcal{B}$ (with $F = F_{\mathcal{A}} \times F_{\mathcal{B}}$) is (kn)-explorable.*

Proof. If $L(\mathcal{A})$ is finite, it is enough to take $k = |L(\mathcal{A})|$ tokens to witness explorability: for each $u \in L(\mathcal{A})$, the token t_u assumes that the input word is u and follows an accepting run of $\mathcal{A}$ over u as long as input letters are compatible with u. As soon as an input letter is not compatible with u, the token t_u is discarded and behaves arbitrarily for the rest of the play.

If $\mathcal{A}$ is k-explorable and $n \geq k$, then Determiniser can win the n-explorability game by using the same strategy with the first k tokens and making arbitrary choices with the $n-k$ remaining tokens.

If $\mathcal{A}$ and $\mathcal{B}$ are k- and n-explorable respectively, then Determiniser can use both strategies simultaneously with $k+n$ tokens in $\mathcal{A} \cup \mathcal{B}$, using k tokens in $\mathcal{A}$ and n tokens in $\mathcal{B}$. If the input word is in $\mathcal{A}$ (resp. $\mathcal{B}$), then the tokens playing in $\mathcal{A}$ (resp. $\mathcal{B}$) will win the play.

In the union product $\mathcal{A} \times \mathcal{B}$, it is enough to take $\max(k, n)$ tokens: if $0 \leq i < \min(k, n)$, the token number i follows the strategy of the token i in $\mathcal{A}$ on the first coordinate, and the strategy of the token i in $\mathcal{B}$ in the second one. If $\min(k, n) \leq i < \max(k, n)$, say wlog $k \leq i < n$, the token i follows an arbitrary strategy on the $\mathcal{A}$-component and the strategy of token i on the $\mathcal{B}$-component.

However, Determiniser may need up to kn tokens to play in $\mathcal{A} \times \mathcal{B}$ when the accepting set is $F_{\mathcal{A}} \times F_{\mathcal{B}}$: the token (i,j) will use the strategy of the token i in the k-explorability game of $\mathcal{A}$ together with the strategy of the token j in the n-explorability game of $\mathcal{B}$. This lower bound of kn cannot be improved: consider for instance $\mathcal{A}_k \times \mathcal{A}_n$, where $\mathcal{A}_k, \mathcal{A}_n$ are from Example 2. ◀

Notice that a similar notion was introduced in [15] under the name *width*. In [15], the emphasis is put on another version of the explorability game, where tokens can be duplicated, and $|Q|$ is an upper bound for the number of necessary tokens. In this work we will on the contrary focus on non-duplicable tokens, for which some results of [15] already apply. In particular the following holds:

▶ **Theorem 5** ([15, Rem. 6.9]). *Given an NFA $\mathcal{A}$ and an integer k, it is* ExpTime*-complete to decide whether $\mathcal{A}$ is k-explorable (even if we fix $k = |Q_{\mathcal{A}}|/2$).*

We aim here at answering a different question:

▶ **Definition 6** (Explorability problem). *The explorability problem is the question, given a non-deterministic automaton $\mathcal{A}$, of deciding whether it is explorable.*

Questions : Is the explorability problem decidable ? If yes, what is its complexity ?
We will first give some motivation for this problem in Section 2.3.

2.3 Link with GFG automata

An automaton $\mathcal{A}$ is Good-for-Games (GFG) if it is 1-explorable, *i.e.* if there is a strategy $\sigma : \Sigma^* \to Q$ resolving the non-determinism based on the word read so far, with the guarantee that the run piloted by this strategy is accepting whenever the input word is in $L(\mathcal{A})$. See *e.g.* [3] for an introduction to GFG automata.

We will give here an additional and stronger link between explorable and GFG automata. In this part we will mainly be interested in automata on infinite words.

One of the main open problems related to GFG automata on infinite words is to decide, given a nondeterministic parity automaton, whether it is GFG. For now, the problem is only known to be in PTime for co-Büchi [16] and Büchi [1] automata. Extending this result even to 3 parity ranks is still open, and only a naive ExpTime upper bound [13] is known in this case. The following result shows that explorability is relevant in this context:

▶ **Theorem 7.** *Given an explorable parity automaton $\mathcal{A}$ of fixed parity index, it is in* PTime *to decide whether it is GFG.*

This is one of the motivations to get a better understanding of explorable automata. Indeed, if we can obtain an efficient algorithm for recognizing them, or if we are in a context guaranteeing that we are only dealing with explorable automata, this result shows that we can obtain an efficient algorithm for recognizing GFG automata.

The rest of this section will be devoted to give a proof sketch of Theorem 7. See Appendix A.1 for formal details. The proof idea is inspired by [1].

Let $\mathcal{A}$ be an explorable parity automaton, of fixed parity index $[i,j]$.

We briefly recall the definition of the game $G_k(\mathcal{A})$ defined in [1], for an arbitrary $k \in \mathbb{N}$. At each round, Adam plays a letter $a \in \Sigma$, then Eve moves her token according to an a-transition, and finally Adam moves his k tokens according to a-transitions. Eve wins the play if her token builds an accepting run, of if all of Adam's tokens build a rejecting run.

We will prove that the game $G_2(\mathcal{A})$ is won by Eve if and only $\mathcal{A}$ is GFG. Since $G_2(\mathcal{A})$ can be solved in PTIME for fixed parity index [1], this is enough to conclude.

First, it is clear that if $\mathcal{A}$ is GFG, then Eve wins $G_2(\mathcal{A})$ [1]: Eve can simply play her GFG strategy with her token, ignoring Adam's tokens.

For the converse, assume Eve wins $G_2(\mathcal{A})$, we want to prove that $\mathcal{A}$ is GFG. We use the following lemma:

▶ **Lemma 8** ([1, Thm. 14]). *Eve wins $G_2(\mathcal{A})$ if and only if Eve wins $G_k(\mathcal{A})$ for all $k \geq 2$.*

Since $\mathcal{A}$ is explorable, there is $k \in \mathbb{N}$ such that $\mathcal{A}$ is k-explorable. Let τ_k be a winning strategy for Determiniser in the k-explorability game of $\mathcal{A}$, and σ_k be a winning strategy for Eve in $G_k(\mathcal{A})$. We show that we can combine these two strategies to yield a GFG strategy σ for $\mathcal{A}$. This proof follows the same idea as in [1] where the explorability hypothesis is not available, but $\mathcal{A}$ is assumed to be Büchi. The strategy σ will store k virtual tokens in its memory. When the automaton reads a new letter $a \in \Sigma$, these k tokens will be updated according to τ_k. Then the choice of σ will follow the strategy σ_k against these k tokens. Notice that the strategies τ_k and σ_k might use additional memory, but this is completely transparent in this proof scheme. If the input word is in $L(\mathcal{A})$, then by correctness of τ_k, one of the k virtual tokens will accept. Thus, by correctness of σ_k, the run chosen by σ will be accepting. Therefore, σ is a correct GFG strategy, witnessing that $\mathcal{A}$ is GFG. This concludes the proof sketch of Theorem 7.

3 Decidability and complexity of the explorability problem

In this section, we prove that the explorability problem is decidable and EXPTIME-complete.

We start by showing in Section 3.1 decidability of the explorability problem for NFAs using the results of [2] as a black box. This yields an algorithm in 2-EXPTIME. We give in Section 3.2 a polynomial reduction in the other direction, thereby obtaining EXPTIME-hardness of the NFA explorability problem. To obtain a matching upper bound and show EXPTIME-completeness, we use again [2], but this time we must "open the black box" and dig into the technicalities of their EXPTIME algorithm while adapting them to our setting. We do so in Section 3.3, directly treating the more general case of Büchi automata.

3.1 2-ExpTime algorithm via a black box reduction

Let us start by recalling the population control problem (PCP) of [2].

▶ **Definition 9** (k-population game). *Given an NFA $\mathcal{B}$ with a distinguished target state $f \in Q_\mathcal{B}$, and an integer $k \in \mathbb{N}$, the k-population game is played similarly as the k-explorability game, only the winning condition differs: Spoiler wins if the game reaches a position where all tokens are in the state f.*

The PCP asks, given $\mathcal{B}$ and $f \in Q_\mathcal{B}$, whether Spoiler wins the k-population game for all $k \in \mathbb{N}$. Notice that this convention is opposite to explorability, where positive instances are defined via a win of Determiniser. The PCP is shown in [2] to be EXPTIME-complete. We will present here a direct exponential reduction from the explorability problem to the PCP.

Let $\mathcal{A}$ be a NFA. Our goal is to build an exponential NFA $\mathcal{B}$ with a distinguished state f such that $(\mathcal{B}, f)$ is a negative instance of the PCP if and only if $\mathcal{A}$ is explorable.

We choose $Q_\mathcal{B} = (Q_\mathcal{A} \times \mathcal{P}(Q_\mathcal{A})) \uplus \{f, \bot\}$, where $f, \bot$ are fresh sink states. The alphabet of $\mathcal{B}$ will be $\Sigma_\mathcal{B} = \Sigma \uplus \{a_{\text{test}}\}$, where a_{test} is a fresh letter.

The initial state of $\mathcal{B}$ is $q_0^{\mathcal{B}} = (q_0^{\mathcal{A}}, \{q_0^{\mathcal{A}}\})$. Notice that we do not need to specify accepting states in $\mathcal{B}$, as acceptance play no role in the PCP.

We finally define the transitions of $\mathcal{B}$ in the following way:

- $(p, X) \xrightarrow{a} (q, \Delta_{\mathcal{A}}(X, a))$ if $a \in \Sigma$ and $q \in \Delta_{\mathcal{A}}(p, a)$,
- $(p, X) \xrightarrow{a_{\text{test}}} f$ if $p \notin F_{\mathcal{A}}$ and $X \cap F_{\mathcal{A}} \neq \emptyset$.
- $(p, X) \xrightarrow{a_{\text{test}}} \bot$ otherwise.

We aim at proving the following Lemma:

▶ **Lemma 10.** *For any $k \in \mathbb{N}$, $\mathcal{A}$ is k-explorable if and only if Determiniser wins the k-population game on $(\mathcal{B}, f)$.*

Notice that as long as letters of Σ are played, the second component of states of $\mathcal{B}$ evolves deterministically and keeps track of the set of reachable states in $\mathcal{A}$. Moreover, the letter a_{test} also acts deterministically on $Q_{\mathcal{B}}$. Therefore, the only non-determinism to be resolved in $\mathcal{B}$ is how the first component evolves, which amounts to building a run in $\mathcal{A}$. Thus, strategies driving tokens in $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. It now suffices to observe that Spoiler wins the k-population game on $(\mathcal{B}, f)$ if and only if he has a strategy allowing to eventually play a_{test} while all tokens are in a state of the form (q, X) with $q \notin F_{\mathcal{A}}$ and $X \cap F_{\mathcal{A}} \neq \emptyset$. This is equivalent to Spoiler winning the k-explorability game of $\mathcal{A}$, since $X \cap F_{\mathcal{A}} \neq \emptyset$ witnesses that the word played so far is in $L(\mathcal{A})$.

This concludes the proof that $\mathcal{A}$ is explorable if and only if $(\mathcal{B}, f)$ is a negative instance of the PCP. So given a NFA $\mathcal{A}$ that we want to test for explorability, it suffices to build $(\mathcal{B}, f)$ as above, and use the ExpTime algorithm from [2] as a black box on $(\mathcal{B}, f)$. Since $\mathcal{B}$ is of exponential size compared to $\mathcal{A}$, we obtain the following result:

▶ **Theorem 11.** *The NFA explorability problem is decidable and in 2-ExpTime.*

3.2 ExpTime-hardness of NFA explorability

We will perform here an encoding in the converse direction: starting from an instance $(\mathcal{B}, f)$ of the PCP, we build polynomially a NFA $\mathcal{A}$ such that $\mathcal{A}$ is explorable if and only if $(\mathcal{B}, f)$ is a negative instance of the PCP.

It is stated in [2] that without loss of generality, we can consider that f is a sink state in $\mathcal{B}$, and we will use this assumption here.

Let $\mathcal{C}$ be the 4-state automaton of Example 2, that is non-explorable and accepts all words on alphabet $\Sigma_{\mathcal{C}} = \{a, b\}$. Recall that as an instance of the PCP, $\mathcal{B}$ does not come with an acceptance condition. We will consider that its accepting set is $F_{\mathcal{B}} = Q_{\mathcal{B}} \setminus \{f\}$.

We will take for $\mathcal{A}$ the product automaton $\mathcal{B} \times \mathcal{C}$ on alphabet $\Sigma_A = \Sigma_{\mathcal{B}} \times \Sigma_{\mathcal{C}}$, with the union acceptance condition: $\mathcal{A}$ accepts whenever one of its components accepts. The transitions of $\mathcal{A}$ are defined as expected: $(p, p') \xrightarrow{a_1, a_2} (q, q')$ in $\mathcal{A}$ whenever $p \xrightarrow{a_1} q$ in $\mathcal{B}$ and $p' \xrightarrow{a_2} q'$ in $\mathcal{C}$.

Since $L(\mathcal{C}) = (\Sigma_{\mathcal{C}})^*$, we have $L(\mathcal{A}) = (\Sigma_{\mathcal{A}})^*$. The intuition for the role of $\mathcal{C}$ in this construction is the following: it allows us to modify $\mathcal{B}$ in order to accept all words, without interfering with its explorability status.

We claim that for any $k \in \mathbb{N}$, $\mathcal{A}$ is k-explorable if and only if Determiniser wins the k-population game on $(\mathcal{B}, f)$.

Assume that $\mathcal{A}$ is k-explorable, via a strategy σ. Then Determiniser can play in the k-population game on $(\mathcal{B}, f)$ using σ as a guide. In order to simulate σ, one must feed to it letters from $\Sigma_{\mathcal{C}}$ in addition to letters from $\Sigma_{\mathcal{B}}$ chosen by Spoiler. This is done by applying a winning strategy for Spoiler in the k-explorability game of $\mathcal{C}$. Assume for contradiction that at some point, this strategy σ reaches a position where all tokens are in a state of the form (f, q) with $q \in Q_{\mathcal{C}}$. Since f is a sink state, when the play continues it will eventually reach a point where all tokens are in (f, q_3), where q_3 is the rejecting sink of $\mathcal{C}$. This is because we are playing letters from $\Sigma_{\mathcal{C}}$ according to a winning strategy for Spoiler in the k-explorability game of $\mathcal{C}$, and this strategy guarantees that all tokens eventually reach q_3 in $\mathcal{C}$. But this state (f, q_3) is rejecting in $\mathcal{A}$, and $L(\mathcal{A}) = (\Sigma_{\mathcal{A}})^*$, so this is a losing position for Determiniser in the k-explorability game of $\mathcal{A}$. Since we assumed σ is a winning strategy in this game, we reach a contradiction. This means that following this strategy σ together with an appropriate choice for letters from $\Sigma_{\mathcal{C}}$, we guarantee that at least one token never reaches the sink state f on its $\mathcal{B}$-component. This corresponds to Determiniser winning in the k-population game on $(\mathcal{B}, f)$.

Conversely, assume that Determiniser wins in the k-population game on $(\mathcal{B}, f)$, via a strategy σ. The same strategy can be used in the k-explorability game of $\mathcal{A}$, by making arbitrary choices on the $\mathcal{C}$ component. As before, this corresponds to a winning strategy in the k-explorability game of $\mathcal{A}$, since there is always at least one token with $\mathcal{B}$-component in $F_{\mathcal{B}} = Q_{\mathcal{B}} \setminus \{f\}$. This achieves the hardness reduction, and allows us to conclude:

▶ **Theorem 12.** *The NFA explorability problem is* ExpTime*-hard.*

▶ Remark 13. Using standard padding arguments, it is straightforward to extend Theorem 12 to ExpTime-hardness of explorability for automata on infinite words, using any of the acceptance conditions defined in Section 2.1.

Let us give some intuition on why we can obtain a polynomial reduction in one direction, but did not manage to do so in the other direction. Intuitively, the explorability problem is "more difficult" than the PCP for the following reason. In the PCP, Spoiler is allowed to play any letters, and the winning condition just depends on the current position. On the contrary, the winning condition of the k-explorability game mentions that the word chosen by Spoiler must belong to the language of the NFA. In order to verify this, we a priori need to append to the arena an exponential deterministic automaton for this language, and this is what is done in Section 3.1. This complicated winning condition is also the source of difficulty of recognizing GFG parity automata.

3.3 ExpTime algorithm for Büchi explorability

▶ **Theorem 14.** *The explorability problem can be solved in* ExpTime *for Büchi automata (and all simpler conditions).*

Due to space constraints we will only sketch the proof of Theorem 14 here. A more detailed account is given in Appendix A.2.

The algorithm is adapted from the ExpTime algorithm for the PCP from [2]. We will recall here the main ideas of this algorithm, and describe how we adapt it to our setting.

Let $\mathcal{A}$ be an NFA, together with a target state f. The idea in [2] is to abstract the population game with arbitrary many tokens by a game called the *capacity game*. This game allows Determiniser to describe only the support of his set of tokens, *i.e.* the set of states occupied by tokens. The sequence of states obtained in a play can be analyzed via a notion of *bounded capacity*, in order to detect whether it actually corresponds to a play with finitely many tokens. This notion can be approximated by the more relaxed *finite capacity*, which is a regular property that is equivalent to bounded capacity in a context where games are finite-memory determined. This property of finite capacity can be verified by a deterministic parity automaton, yielding a parity game that can be won by Spoiler if and only if $(\mathcal{A}, f)$ is a positive instance of the PCP. Since this parity game has size exponential in $\mathcal{A}$, this yields an ExpTime algorithm for the PCP.

Here, we will perform the following tweaks to this construction. We now start with a Büchi automaton $\mathcal{A}$, and want to decide whether it is explorable.

First, we need to control that the infinite word played by Spoiler is in $L(\mathcal{A})$. This requires to build a deterministic parity automaton $\mathcal{D}$ recognising $L(\mathcal{A})$, and incorporate it into the arena. The size of $\mathcal{D}$ is exponential with respect to $\mathcal{A}$. We then follow [2] and build the capacity game augmented with $\mathcal{D}$. This time, a sequence of supports is winning if infinitely many of them contain an accepting state. We emphasize that we use here a particularity of the Büchi condition: observing the sequences of support sets of tokens is enough to decide whether one of the tokens follows an accepting run. The same particularity was used in [1], and was a crucial tool allowing to give a PTime algorithm for Büchi GFGNess. Since this modification still allows us to manipulate supports as simple sets, we can make use of the capacity game as before. We give in Appendix A.2, Remark 39 an example showing that a naive adaptation of this construction to co-Büchi automata would not be correct.

Finally, we show that we can as in [2] obtain a parity game of exponential size characterizing explorability of $\mathcal{A}$, yielding the wanted ExpTime algorithm.

We also remark that as in [2], this construction gives a doubly exponential upper bound on the number of tokens needed to witness explorability. Moreover, the proof from [2] that this is tight also stands here.

4 Explorability with countably many tokens

In this section we look at the same problem of explorability of an automaton, but we now allow for infinitely many tokens. More precisely, we will redefine the explorability game to allow an arbitrary cardinal for the number of tokens, then consider decidability problems regarding that game. This notion will mainly be interesting for automata on infinite words.

4.1 Definition and basic results

The following definition extends the notion of k-explorability to non-integer cardinals:

▶ **Definition 15** (κ-explorability game)**.** *Consider an automaton $\mathcal{A}$ and a cardinal κ. The κ-explorability game on $\mathcal{A}$ is played on the arena $(Q_{\mathcal{A}})^{\kappa}$, between Determiniser and Spoiler. They play as follows.*

- *The initial position is S_0 associating q_0 to all κ tokens.*
- *At step i, from position S_{i-1}, Spoiler chooses a letter $a_i \in \Sigma$, and Determiniser chooses S_i such that for any token α, $S_{i-1}(\alpha) \xrightarrow{a} S_i(\alpha)$ is a transition in $\mathcal{A}$.*

The play is won by Determiniser if for any $\beta \leq \omega$ such that the word $(a_i)_{1 \leq i < \beta}$ is in $\mathcal{L}(\mathcal{A})$, there is a token $\alpha \in \kappa$ building an accepting run, meaning that the sequence $(S_i(\alpha))_{i<\beta}$ is an accepting run. Otherwise the winner is Spoiler.

We will say in particular that $\mathcal{A}$ is ω-explorable if Determiniser wins the game with ω tokens. We use here the notation ω for convenience, it should be understood as the countably infinite cardinal $\aleph_0$. We will however explicitly use the fact that such an amount of tokens can be labelled by $\mathbb{N}$, in order to describe strategies for Spoiler or Determiniser in the ω-explorability game. The following lemma gives a first few results on generalised explorability.

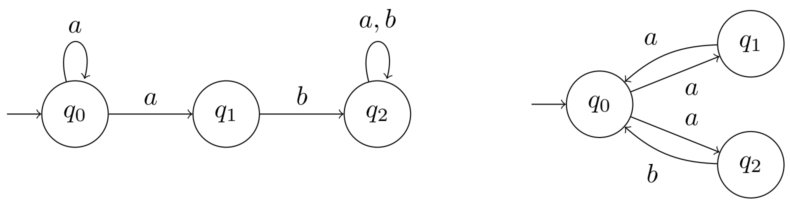


Figure 1 Two safety automata. Left: ω-explorable but not explorable. Right: not ω-explorable.

▶ **Lemma 16.**

- *Determiniser wins the explorability game on $\mathcal{A}$ with $|\mathcal{L}(\mathcal{A})|$ tokens.*
- *ω-explorability is not equivalent to explorability*
- *There are non ω-explorable safety automata.*

Proof. For the first item, a strategy for Determiniser is to associate a token to each word of $\mathcal{L}(\mathcal{A})$ and to have it follow an accepting run for that word. Let us add a few details on the cardinality of $L(\mathcal{A})$. First, a dichotomy result has been shown in [19] (even in the more general case of infinite trees): if $L(\mathcal{A})$ is not countable, then it has the cardinality of continuum, and this happens if and only if $L(\mathcal{A})$ contains a non-regular word. In this case, we can simply associate a token with every possible run. In the other case where $L(\mathcal{A})$ is countable, we have to associate an accepting run to each word, and this can be done without needing the Axiom of Countable Choice: a canonical run can be selected (*e.g.* lexicographically minimal).

We now want to prove that there are automata that are ω-explorable but not explorable. One such automaton is given in Figure 1 (left), where the rejecting sink state is omitted. Against any finite number of tokens, Spoiler has a strategy to eliminate them one by one, by playing a while Determiniser sends tokens to q_1, and b the first time q_1 is empty after the play of Determiniser. On the other hand, with tokens indexed by ω, Determiniser can keep the token 0 in q_0, and send token i to q_1 at step i. Those strategies are winning, which proves both non explorability and ω explorability of the automaton.

The last item is proven by the second example from Figure 1. A winning strategy for Spoiler against countable tokens consists in labelling the tokens with integers, then targeting each token one by one (first token 0, then 1, 2, *etc.*). Each token is removed using the correct two-letters sequence (a, then b if the token is in q_1 or a if it is in q_2). With this strategy, every token is removed at some point, even if there might always be tokens in the game. ◀

The first item of Lemma 16 implies that the ω-explorability game only gets interesting when we look at automata over infinite words: since any language of finite words over a finite alphabet is countable, Determiniser wins the corresponding ω-explorability game. We will therefore focus on infinite words in the following.

Let us emphasize the following slightly counter-intuitive fact: in the ω-explorability game, it is always possible for Determiniser to guarantee that infinitely many tokens occupy each currently reachable state. However, even in a safety automaton, this is not enough to win the game, as it does not prevent that each individual token might be eventually "killed" at some point. As the following Lemma shows, this phenomenon does not occur in reachability automata.

▶ **Lemma 17.** *Any reachability automaton is ω-explorable.*

Proof. For every $w \in \Sigma^*$ such that there is a finite run ρ leading to an accepting state, Determiniser can use a single token following ρ. This token will accept all words of $w \cdot \Sigma^\omega$. Since Σ^* is countable, we only need countably many such tokens to cover the whole language, hence the result.

Let us give another equally simple view: a winning strategy for Determiniser in the ω-explorability game is to keep infinitely many tokens in each currently reachable state, as described above. Since acceptance in a reachability automaton is witnessed at a finite time, this strategy is winning. ◀

4.2 ExpTime algorithm for co-Büchi automata

We already know, from the example of Figure 1, that the result from Lemma 17 does not hold in the case of safety automata. However we have the following decidability result, which talks about co-Büchi automata, and therefore still holds for safety automata as a subclass of co-Büchi.

▶ **Theorem 18.** *The ω-explorability of co-Büchi automata is decidable in* ExpTime.

To prove this result, we will use the following *elimination game*. $\mathcal{A}$ will from here on correspond to a co-Büchi (complete) automaton. We start by building a deterministic co-Büchi automaton $\mathcal{D}$ for $L(\mathcal{A})$ (*e.g.* using the breakpoint construction [18]).

▶ **Definition 19** (Elimination game). *The elimination game is played on the arena $\mathcal{P}(Q_\mathcal{A}) \times Q_\mathcal{A} \times Q_\mathcal{D}$. The two players are named Protector and Eliminator, and the game proceeds as follows, starting in the position $(\{q_0^\mathcal{A}\}, q_0^\mathcal{A}, q_0^\mathcal{D})$.*

- *From position (B, q, p) Eliminator chooses a letter $a \in \Sigma$.*
- *If q is not a co-Büchi state, Protector picks a state $q' \in \Delta_\mathcal{A}(q, a)$.*
- *If q is a co-Büchi state, Protector picks any state $q' \in \Delta_\mathcal{A}(B, a)$. Such an event is called* elimination*.*
- *The play moves to position $(\Delta_\mathcal{A}(B, a), q', \delta_\mathcal{D}(p, a))$.*

Such a play can be written $(B_0, q_0, p_0) \xrightarrow{a_1} (B_1, q_1, p_1) \xrightarrow{a_2} (B_2, q_2, p_2) \ldots$, and Eliminator wins if infinitely many q_i and finitely many p_i are co-Büchi states.

Intuitively, what is happening in this game is that Protector is placing a token that he wants to protect in a reachable state, and Eliminator aims at bringing that token to a co-Büchi state while playing a word of $L(\mathcal{A})$. If Protector eventually manages to preserve his token from elimination on an infinite suffix of the play, he wins.

▶ **Lemma 20.** *The elimination game can be solved in polynomial time (in the size of the game).*

Proof. To prove this result, we simply need to note that the winning condition is a parity condition of fixed index. If we label the co-Büchi states q_i with rank 1, the co-Büchi states p_i with rank 2, and the others with 3, then take the lowest rank in (B_i, q_i, p_i) (ignoring B_i), Eliminator wins if and only if the inferior limit of ranks is even. As any parity game with 3 ranks can be solved in polynomial time [7], this is enough to get the result. ◀

We want to prove the equivalence between this game and the ω-explorability game to obtain Theorem 18.

▶ **Lemma 21.** *$\mathcal{A}$ is ω-explorable if and only if Protector wins the elimination game on $\mathcal{A}$.*

Proof. First let us suppose that Eliminator wins the elimination game on $\mathcal{A}$. To build a strategy for Spoiler in the ω-explorability game of $\mathcal{A}$, we first take a function $f : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|f^{-1}(n)|$ is infinite (for instance f is described by the sequence $0,\ 0,1,\ 0,1,2,\ 0,1,2,3,\ldots$). The strategy for Spoiler will focus on sending token $f(0)$, then $f(1)$, then $f(2)$, *etc.* to a co-Büchi state.

Let σ be a memoryless winning strategy for Eliminator in the elimination game (recall that parity games do not require memory [12]). Spoiler will follow this strategy σ in the ω-explorability game, by keeping an imaginary play of the elimination game in his memory: $M = \mathcal{P}(Q_\mathcal{A}) \times Q_\mathcal{A} \times Q_\mathcal{D} \times \mathbb{N}$.

- At first the memory holds the initial state $(\{q_0^\mathcal{A}\}, q_0^\mathcal{A}, q_0^\mathcal{D}, 0)$, and the current target is given by the last component: it is the token $f(0)$.
- From (B, q, p, n) Spoiler plays in both games the letter a given by σ.
- Once Determiniser has played, Spoiler moves the memory to $(\Delta_\mathcal{A}(B, a), q', \delta_\mathcal{D}(p, a), n)$ where q' is the new position of the token $f(n)$, except if q was a co-Büchi state, in which case we move to $(\Delta_\mathcal{A}(B, a), q', \delta_\mathcal{D}(p, a), n + 1)$ where q' is the new position of the token $f(n + 1)$. We then go back to the previous step.

This strategy builds a play of the elimination game in the memory, that is consistent with σ. We know that σ is winning, which implies that the word played is in $\mathcal{L}(\mathcal{A})$, and that every $n \in \mathbb{N}$ is visited (each elimination increments n, and there are infinitely many of those). An elimination happening while the target is the token $f(n)$ corresponds, on the exploration game, to that token visiting a co-Büchi state. Ultimately this means that Determiniser did not provide any accepting run, while Spoiler did play a word from $\mathcal{L}(\mathcal{A})$, and therefore won.

Let us now consider the situation where Protector wins the elimination game, using some strategy τ. We want to build a winning strategy for Determiniser in the ω-explorability game. Similarly, this strategy will keep track of a play in the elimination game in its memory. Determiniser will maintain ω tokens in any reachable state, while focusing on a particular token which follows the path of the current target in the elimination game. When that token visits a co-Büchi state, we switch to the new token specified by τ.

Since τ is winning in the elimination game, either the word played by Spoiler is not in $\mathcal{L}(\mathcal{A})$, which ensures a win for Determiniser, or there are no eliminations after some point, meaning that the target token at that point never visits another co-Büchi state, which also implies that Determiniser wins. ◀

With Lemmas 20 and 21 we get a proof of Theorem 18, since the elimination game associated to $\mathcal{A}$ is of exponential size and can be built using exponential time.

4.3 ExpTime-hardness of the ω-explorability problem

▶ **Theorem 22.** *The ω-explorability problem for (any automaton model embedding) safety automata is* ExpTime*-hard.*

We give a quick summary of the proof in this section. The full proof can be found in Appendix A.3. The main idea will be to reduce the acceptance problem of a PSpace alternating Turing machine (ATM) to the ω-explorability problem of some automaton that we build from the machine. This reduction is an adaptation of the one from [2] showing ExpTime-hardness of the NFA population control problem (defined in Section 3.1).

The computation of an ATM can be seen as a game between two players who respectively aim for acceptance and rejection of the input. These players influence the output by choosing the transitions when facing a non-deterministic choice, that can belong to either one of them.

Let us first describe the automaton built in [2]. In that reduction, the choices made by the ATM players are translated into choices for Determiniser and Spoiler. The automaton has two main blocks: one dedicated to keeping track of the machine's configuration, which we call Config, and another focusing on the simulation of the ATM choices, which we call Choices. In Config, there is no non-determinism: the tokens move following the transitions of the machine given as input to the automaton. In Choices, Determiniser can pick a transition by sending his token to the corresponding state, while Spoiler uses letters to pick his.

The automaton constructed this way will basically read a sequence of runs of the ATM. At each run, some tokens must be sent into both blocks. Reaching an accepting state of a run lets Spoiler send some tokens from Choices to his target state, specifically those whose choices for the transitions of the ATM were followed. He can then restart with the remaining tokens until all are in the target. This process will ensure a win for Spoiler if he has a winning strategy in the ATM game. If he does not, then Determiniser can use a strategy ensuring rejection in the ATM game to avoid the configurations where he loses tokens, provided he starts with enough tokens.

This equivalence between acceptance of the ATM and the automaton being a positive instance of the PCP provides the ExpTime-hardness of their problem.

In our setup, getting rid of tokens one by one is not enough: Spoiler needs to be able to target a specific token and send it to the target state (which is now the rejecting state $\perp$) in one run. If he can do that, repeating the process for every token, without omitting any, ensures his win. If he cannot, then Determiniser has a strategy to pick a specific token and preserving it from $\perp$, and therefore wins.

This is why we adapt our reduction to allow Spoiler to target a specific token, no matter where it chooses to go. To do so, we change the transitions so that winning a run lets Spoiler additionally send every token from Config into $\perp$. With that and the fact that he can already target a token in Choices, we get a winning strategy for Spoiler when the ATM is accepting.

If the ATM is rejecting, Spoiler is still able to send some tokens to $\perp$, but he no longer has that targeting ability, which is how Determiniser is able to build a strategy preserving a specific token to win. To ensure the sustainability of this method, Determiniser needs to keep ω additional tokens following his designated token, so that he always has ω tokens to spread into the gadgets every time a new run starts.

Overall, we are able to compute in polynomial time from the ATM a safety automaton that is ω-explorable if and only if the ATM rejects its input. Since acceptance of a polynomial space ATM is known to be ExpTime-hard, we obtain Theorem 22.

Conclusion

We introduced and studied the notions of explorability and ω-explorability, for automata on finite and infinite words. We showed that these problems are ExpTime-complete for Büchi condition in the first case and co-Büchi condition in the second case.

It is plausible that these results could be generalised to higher parity conditions, for instance by replacing the notion of support set by Safra trees, but this is outside of the scope of this paper and we leave this investigation for further research.

Although we showed that the original motivation of using explorability to improve the current knowledge on the complexity of the GFGness problem for all parity automata cannot be directly achieved, since deciding explorability is at least as hard as GFGness, we believe that explorability is a natural property in the study of degrees of nondeterminism, and that this notion could be used in other contexts as a middle ground between deterministic and non-deterministic automata.

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A Appendix

A.1 Link with GFG automata

We describe here in more detail how, assuming that $\mathcal{A}$ is explorable and Eve wins $G_k(\mathcal{A})$ for some $k \in \mathbb{N}$, we obtain a GFG strategy for $\mathcal{A}$. Let us note $Q = Q_{\mathcal{A}}$ the set of states of $\mathcal{A}$.

In the proof sketch of Section 2.3, we defined τ_k to be a winning strategy for Determiniser in the k-explorability game, and σ_k a winning strategy for Eve in $G_k(\mathcal{A})$.

Let us explicit in detail the shape of these strategies. The strategy τ_k has access to the history of the play in the k-explorability game, and must decide on a move for Determiniser. Notice that it is always enough to know the history of the opponent's moves (here the letters of Σ played so far), since this allows to compute the answer of Determiniser at each step, and therefore build a unique play. Thus we can take for τ_k a function $\Sigma^* \to Q^k$. If the word played so far is $u \in \Sigma^*$, the tuple of states reached by the k tokens moved according to τ_k is $\tau_k(u) \in Q^k$, with in particular $\tau_k(\varepsilon) = (q_0^{\mathcal{A}}, \dots, q_0^{\mathcal{A}})$.

If $w = a_1a_2 \dots \in \Sigma^\omega$, and $i \in \mathbb{N}$, let us note $(q^i_{w,1}, \dots, q^i_{w,k}) = \tau_k(a_1 \dots a_i)$. That is $q^i_{w,j}$ is the state reached by the j^{th} token after i steps in the run induced by τ_k and u. If $j \in [1,k]$, let us note $\rho_{u,j}$ the infinite run $q^0_{w,j}q^1_{w,j}q^2_{w,j}\dots$, followed by the j^{th} token in this play. By definition of τ_k, we have the guarantee that for all $w \in L(\mathcal{A})$, there exists $j \in [1,k]$ such that $\rho_{w,j}$ is accepting.

If $u = a_1 \dots a_n \in \Sigma^*$ is a finite word, we define $\tau'_k(u) = (\tau_k(\varepsilon), \tau_k(a_1), \tau_k(a_1a_2) \dots, \tau_k(u))$ the list of partial runs induced by τ_k on u.

Let us now turn to the strategy σ_k of Eve in $G_k(\mathcal{A})$. The type of this strategy is $\sigma_k : \Sigma^* \times (Q^k)^* \to Q$. Indeed, this time, the history of Adam's moves must contain his choice of letters together with his choices of positions for his k tokens. So $\sigma_k(u, \gamma)$ gives the state reached by Eve's token after an history (u, γ) for the moves of Adam. Notice that at each step, Eve must move before Adam in this game $G_k(\mathcal{A})$, so γ does not contain the choice of Adam on the last letter of u. This means that except for $u = \varepsilon$, we can always assume $|u| = |\gamma| + 1$ in a history (u, γ).

We have the guarantee that if Adam plays an infinite word w together with runs $\rho_1, \dots, \rho_k$ on w, at least one of which is accepting, then the run yielded by σ_k against $(w, (\rho_1, \dots, \rho_k))$ is accepting.

We finally define the GFG strategy σ for $\mathcal{A}$, of type $\Sigma^* \to Q$, by induction: $\sigma(\varepsilon) = q_0^{\mathcal{A}}$, and $\sigma(ua) = \sigma_k(ua, \tau'_k(u))$.

This amounts to playing the strategy σ_k in $G_k(\mathcal{A})$, against Adam playing a word w and moving his k tokens according to the strategy τ_k against w. If the infinite word $w = a_1a_2 \dots$ chosen by Adam is in $L(\mathcal{A})$, then by correctness of τ_k one of the k runs $\rho_{w,1}, \dots, \rho_{w,k}$ yielded by τ_k is accepting. Hence, by correctness of σ_k, the run $\sigma(\varepsilon)\sigma(a_1)\sigma(a_1a_2)$ yielded by σ (based on σ_k) is accepting. This concludes the proof that σ is a correct GFG strategy for $\mathcal{A}$, witnessing that $\mathcal{A}$ is GFG.

A.2 ExpTime algorithm for Büchi explorability

We will prove here Theorem 14 from Section 3.3.

In this part, $\mathcal{A} = (\Sigma, Q, q_0^{\mathcal{A}}, \Delta_{\mathcal{A}}, F_{\mathcal{A}})$ is a non-deterministic Büchi automaton. We start by computing in exponential time an equivalent deterministic parity automaton $\mathcal{D} = (\Sigma, Q_{\mathcal{D}}, q_0^{\mathcal{D}}, \delta_{\mathcal{D}}, F_{\mathcal{D}})$, via any standard method.

The algorithm described in this section is adapted from [2]. Many results from this previous work still hold in our framework. We will however need to adapt some constructions and give new arguments, both to fit our explorability framework, and to generalize from NFA to Büchi automata.

▶ **Definition 23** (Transfer graph). *A transfer graph G is a subset of $Q \times Q$. We say that it is compatible with a letter a if every edge in G corresponds to a transition in $\mathcal{A}$ labelled by a, i.e. for any $(q,r) \in G$, we have $(q,a,r) \in \Delta_{\mathcal{A}}$. In other words, G is a subgraph of the transition graph of the letter a.*

Given a transfer graph G and a set of states $X \subseteq Q$, we note $G(X) = \{q \in Q \mid \exists r \in X, (q,r) \in G\}$. We call respectively $Dom(G)$ and $Im(G)$ the projections of G on its first and second coordinate, i.e. $Dom(G) = \{q \in Q \mid \exists r \in Q, (q,r) \in G\}$ and $Im(G) = G(Q)$.

The composition of transfer graphs is defined the natural way: $G \cdot H = \{(x,z) \mid \exists y, (x,y) \in G \wedge (y,z) \in H\}$.

▶ **Definition 24** (Support game). *The support game is played in the arena $\mathcal{P}(Q) \times Q_{\mathcal{D}}$, called* support arena*. It is played as follows by Determiniser and Spoiler.*

- *The starting support is $S_0 = (\{q_0^{\mathcal{A}}\}, q_0^{\mathcal{D}})$.*
- *At any given step with support (B,q), Spoiler chooses a letter $a \in \Sigma$, then Determiniser chooses a transfer graph G compatible with a, and with $Dom(G) = B$. The play then moves to $(Im(G), \delta_{\mathcal{D}}(q,a))$.*

A play can be represented by a sequence $(B_0,q_0) \xrightarrow{a_1,G_1} (B_1,q_1) \xrightarrow{a_2,G_2} (B_2,q_2) \dots$.

We say that Spoiler wins the play if the run $q_0q_1q_2\dots$ of $\mathcal{D}$ is parity accepting, while only finitely many B_i contain Büchi states (from $F_{\mathcal{A}}$).

Note that a winning strategy for Determiniser in the support game cannot in general be interpreted as a witness of explorability. This is illustrated by the automaton $\mathcal{C}$ from Example 2. For any $k \in \mathbb{N}$, the k-explorability game is won by Spoiler on that automaton, while Determiniser wins the support game. Intuitively, the support game does not account for the limits of resources for Determiniser.

On the other hand, a winning strategy for Spoiler in this support game does translate into a non-explorability witness, *i.e.* a strategy for Spoiler in the k-explorability game for any k. The support game is therefore "too easy" for Determiniser, and this is what we try to correct in the following.

▶ **Definition 25** (Projection of a play). *Given a play $S_0 \xrightarrow{a_1} S_1 \xrightarrow{a_2} S_2 \dots$ in the k-explorability game, the projection of that play in the support arena is the play $(B_0,q_0) \xrightarrow{a_1,G_1} (B_1,q_1) \xrightarrow{a_2,G_2} (B_2,q_2) \dots$, where:*

- *B_i is the support of S_i (states occupied in S_i),*
- *$q_0 = q_0^{\mathcal{D}}$ and $q_{i+1} = \delta_{\mathcal{D}}(a_{i+1}, q_i)$ for all i,*
- *$G_{i+1} = \{(S_i(j), S_{i+1}(j)) \mid j \in [0, k-1]\}$.*

This corresponds to forgetting the multiplicity of tokens and only keeping track of the transitions that are used.

▶ **Definition 26** (Realisable play). *A play in the support arena is realisable if it is the projection of a play in the k-explorability game for some $k \in \mathbb{N}$.*

We would like to restrict plays in the support arena to realisable ones only. To do so, we define the notion of capacity as follows.

▶ **Definition 27** (Accumulator and capacity [2]). *In a play* $(B_0, q_0) \xrightarrow{a_1,G_1} (B_1, q_1) \xrightarrow{a_2,G_2} (B_2, q_2) \ldots$, *an* accumulator *is a sequence* $(T_j)_{j\in\mathbb{N}}$ *such that for any* j, $T_j \subseteq B_j$ *and* $T_{j+1} \supseteq G_{i+1}(T_j)$. *An edge* $(q, r) \in G_{j+1}$ *is an* entry *for* $(T_j)_{j\in\mathbb{N}}$ *at index* i *if* $q \notin T_j$ *and* $r \in T_{j+1}$.

A play has finite capacity *if every accumulator has finitely many entries, and* bounded capacity *if the number of entries of its accumulators is bounded.*

This definition gives us tools to talk about realisable plays in a more practical way, as shown by the following Lemma. Note that although the explorability game is replaced by the population control game in [2], the same proof still applies here.

▶ **Lemma 28** ([2, Lem 3.5]). *A play is realisable if and only if it has bounded capacity.*

Moreover, the proof of Lemma 28 can also be used to get the following result, which we will use later. Note that we talk about the explorability game in this Lemma, but this only concerns its arena regardless of the winning condition. The proof holds because the arena from [2] is identical.

▶ **Lemma 29** ([2, Lem 3.5]). *If Determiniser has a strategy* τ *in the support arena such that any play compatible with* τ *has capacity bounded by* c*, then he has a strategy* τ' *in the* 2^{c+1}*-tokens explorability game such that any play compatible with* τ' *has its projection compatible with* τ.

We will use the notion of capacity to define the following game, using finite capacity instead of bounded to obtain a winning condition.

▶ **Definition 30** (Capacity game). *The capacity game is played in the support arena. Given a play* $(B_0, q_0) \xrightarrow{a_1,G_1} (B_1, q_1) \xrightarrow{a_2,G_2} (B_2, q_2) \ldots$, *Spoiler wins if it is a winning play in the support game, or if it has infinite capacity.*

▶ **Lemma 31** ([2, Prop 3.8]). *Either Spoiler or Determiniser wins the capacity game, and the winner has a winning strategy with finite memory.*

Proof. Although this result talks about slightly different objects than in [2, Prop 3.8], their proof actually still stand with our definitions of capacity game and support game. The proof proceeds by building a nondeterministic Büchi automaton verifying that the capacity is infinite, determinising it into a parity automaton, and incorporating it into the arena to yield a parity game equivalent to the capacity game. The winner of this parity game has a positional strategy, which corresponds to a finite memory strategy in the capacity game. ◀

▶ **Lemma 32** (adapted from [2, Prop 3.9]). *If Spoiler wins the capacity game, then he wins the* k*-explorability game for any* k.

Proof. Here Spoiler can simply apply the strategy for the capacity game to the explorability game, by remembering only the information that is relevant from the point of view of the capacity game (*i.e.* the supports and transfer graphs).This will simulate a realisable play of the capacity game, which has bounded capacity by Lemma 28. Since the strategy is winning in the capacity game, and this simulated play cannot have infinite capacity, Spoiler wins the underlying support game. This ensures the win for Spoiler in the explorability game: he plays a word of $L(\mathcal{A})$ as witnessed by the acceptance of $\mathcal{D}$, while finitely many Büchi states are witnessed by tokens of Determiniser. We use here the particular property of Büchi condition: one of the tokens follows an accepting run if and only if it occurs infinitely many times that the support set occupied by tokens contains a Büchi state. ◀

▶ **Lemma 33** (adapted from [2, Prop 3.10]). *If Determiniser wins the capacity game using finite memory M, then he wins the k-explorability game for some $k \in \mathbb{N}$.*

Proof. We first prove that under these conditions, Determiniser can win the capacity game while ensuring a capacity bounded by $|M| \times |Q_{\mathcal{D}}| \times 4^{|Q|}$.

Let us consider a winning strategy τ with memory M for Determiniser in the capacity game. We take a play $(B_0, q_0) \xrightarrow{a_1, G_1} (B_1, q_1) \xrightarrow{a_2, G_2} (B_2, q_2) \ldots$ compatible with τ, and we show that its capacity is bounded by $|M| \times |Q_{\mathcal{D}}| \times 4^{|Q|}$.

Given an accumulator $\mathcal{T} = (T_i)_{i \in \mathbb{N}}$, if there are two integers $i < j$ such that $m_i = m_j$ (memory states at steps i and j), $B_i = B_j$, $q_i = q_j$ and $T_i = T_j$, then one can build a play that loops on the corresponding interval, while still being compatible with τ. This accumulator cannot have infinitely many entries, so $\mathcal{T}$ does not have any entry in the interval $[i, j]$. As a consequence, if i and j are entry times, we have $(m_i, B_i, q_i, T_i) \neq (m_j, B_j, q_j, T_j)$, which means there can be at most $|M| \times 2^{|Q|} \times |\mathcal{Q}_{\mathcal{D}}| \times 2^{|Q|} = |M| \times |Q_{\mathcal{D}}| \times 4^{|Q|}$ entries in the accumulator $\mathcal{T}$.

We now know that the capacity of any play compatible with τ is bounded by $|M| \times |Q_{\mathcal{D}}| \times 4^{|Q|}$. Take $k = 2^{1+|M| \times |Q_{\mathcal{D}}| \times 4^{|Q|}}$. Lemma 29 then provides a strategy for Determiniser in the k-explorability game, that ensures that the successive supports (*i.e.* the sets of states occupied by tokens) contain Büchi states infinitely often. This means that at least one token visits Büchi states infinitely often, since there are finitely many tokens. This ensures a win for Determiniser. ◀

These Lemmas 32 and 33 give a way to solve the explorability problem if we can efficiently find the winner of the corresponding capacity game. Note that we could use the parity game built in the proof of Lemma 31 to solve the problem, but this would yield a doubly exponential algorithm since the parity automaton that we build in this proof is itself doubly exponential.

The following gives an exponential time algorithm for solving the capacity game, and therefore the explorability problem.

▶ **Definition 34** (Leaks and separations). *If G and H are two transfer graphs, we say that G leaks at H if there are three states q, x, y such that $(q, y) \in G \cdot H$, $(x, y) \in H$ and $(q, x) \notin G$.*

We say that G separates states r and t if there is a q such that $(q, r) \in G$ and $(q, t) \notin G$. The separator of G, noted $\mathtt{Sep}(G)$, is the set of all such (r, t).

Note that in a play denoted as before, whenever $i < j < n$, we have $\mathtt{Sep}(G[i, n]) \subseteq \mathtt{Sep}(G[j, n])$.

We will now define the tracking list of a play. The point of that list will be to provide an easy way to detect indices that leak infinitely often.

▶ **Definition 35** (Tracking list). *The tracking list $\mathcal{L}_n$ at step n is a list of transfer graphs $\{G[i_1, n], \ldots, G[i_{k_n}, n]\}$. It is defined inductively, with $\mathcal{L}_0$ the empty list, and $\mathcal{L}_n$ computed as follows.*

- *We update every $G[i, n-1]$ in $\mathcal{L}_{n-1}$ into $G[i, n]$ by composing with G_n.*
- *We then add $G[n-1, n] = G_n$ at the end of the list.*
- *And finally we clean the list, by removing any graph with a separator identical to the previous one.*

If for some i, $G[i, n] \in \mathcal{L}_n$ for every $n > i$, we say that i is remanent.

To properly use these tracking lists, it suffices to know that the following result holds. For more details we refer the to [2].

▶ **Lemma 36** ([2, Lem 4.4]). *A play has infinite capacity if and only if there is a remanent index that leaks infinitely often.*

We now define a game $\mathcal{G}_{\mathcal{A}}$ associated to $\mathcal{A}$, that extends the support arena using tracking lists to detect infinite capacity plays. Once again, this is an adaptation from [2].

The **states** of $\mathcal{G}_{\mathcal{A}}$ are in $\mathcal{P}(Q) \times Q_{\mathcal{D}} \times \mathcal{G}^{\leq |Q|^2}$, where $\mathcal{G}^{\leq |Q|^2}$ is the set of lists of at most $|Q|^2$ transfer graphs. Each state can be written as (B, q, L) where B is a subset of Q, q is a state of $\mathcal{D}$, and L is a tracking list. The initial state is $(\{q_0^{\mathcal{A}}\}, q_0^{\mathcal{D}}, \varepsilon)$.

The **transitions** are the ones that can be written $(B, q, L) \xrightarrow{p,a,G} (B', q', L')$ with the following conditions.

- $(B, q) \xrightarrow{a,G} (B', q')$ is a transition from the support arena.
- L' is obtained by updating L with G, as detailed in the definition of tracking list.
- Take $L = \{H_1, \ldots, H_k\}$ and $L' = \{H'_1, \ldots, H'_{k'}\}$. Let p' be the smallest index such that $H_{p'}$ leaks at G, or $k + 1$ if there is no such index. Let p'' be the smallest index such that $H'_{p''} \neq H_{p''} \cdot G$, or $k + 1$ if there is none. We then take $p = \min(2p' + 1, 2p'')$ (which implies that $p \in [2, 2|Q|^2 + 1]$).

To choose a transition, Spoiler first chooses a letter, then Determiniser picks a transition graph compatible with that letter. The rest is determined by the conditions above. This creates a play that can be denoted as $(B_0, q_0, L_0) \xrightarrow{a_1,G_1,p_1} (B_1, q_1, L_1) \xrightarrow{a_2,G_2,p_2} \ldots$.

The winning condition for Spoiler goes as follows. Either the inferior limit of $(p_i)_{i>0}$ is odd, or the run $(q_i)_{i\geq 0}$ is accepting while there are finitely many accepting states seen in $(B_i)_{i\geq 0}$.

▶ **Lemma 37** (adapted from [2, Thm 4.5]). *Spoiler wins $\mathcal{G}_{\mathcal{A}}$ if and only if he wins the capacity game.*

Proof. First note that strategies in the support arena can be easily translated to $\mathcal{G}_{\mathcal{A}}$ and conversely, since in both cases Spoiler only chooses letters while Determiniser picks transfer graphs, and the rest is determined by these data.

If Spoiler has a winning strategy in $\mathcal{G}_{\mathcal{A}}$, then he can play the same strategy in the capacity game. Such a play can be written as $(B_0, q_0) \xrightarrow{a_1,G_1} (B_1, q_1) \xrightarrow{a_2,G_2} \ldots$, and the play of $\mathcal{G}_{\mathcal{A}}$ happening in the memory of Spoiler is $(B_0, q_0, L_0) \xrightarrow{a_1,G_1,p_1} (B_1, q_1, L_1) \xrightarrow{a_2,G_2,p_2} \ldots$. We use the notation $L_n = \{H_n^1, \ldots, H_n^{k_n}\}$.

Since Spoiler plays according to a winning strategy in the simulated game $\mathcal{G}_{\mathcal{A}}$, at least one of his winning conditions for that game hold in this play.

If the limit parity is $2p + 1$ for some p, then for any n large enough, H_n^p is the same as H_n^{p+1} (otherwise there would be a parity less than $2p + 1$ later) and leaks infinitely often, so Spoiler wins the capacity game.

If the run $(q_i)_{i\geq 0}$ is accepting while there are finitely many accepting states seen in $(B_i)_{i\geq 0}$, then this also ensures the win for Spoiler in the capacity game.

In both cases, the play is therefore won by Spoiler.

On the other hand, if Spoiler wins the capacity game, he can also use the same strategy in $\mathcal{G}_{\mathcal{A}}$, with the same correspondence between the winning conditions.

◀

We can finally conclude with the main result of this section:

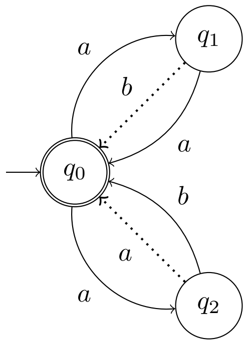

Figure 2 A co-Büchi automaton on which the projection of a play in not enough to determine the winner (the dotted lines represent co-Büchi transitions)

▶ **Theorem.** *The Büchi explorability problem can be solved in* ExpTime.

Proof. To prove this result, it is enough to prove that the game $\mathcal{G}_\mathcal{A}$ can be solved in exponential time in the size of $\mathcal{A}$, since the answer to that problem also answers the explorability of $\mathcal{A}$. We show that the winning condition of the game $\mathcal{G}_\mathcal{A}$ for Spoiler can be seen as a disjunction of parity conditions. Formally, it is of the form Parity$\vee$(Parity$\wedge$ Co-Büchi). But it is straightforward to turn the second disjunct into a parity condition with twice as many priorities. Thus $\mathcal{G}_\mathcal{A}$ can be seen as a generalised parity game. Such games are studied in [8], which gives us an algorithm for solving $\mathcal{G}_\mathcal{A}$ in time $O(m^{4d}m^2)\frac{(2d)!}{d!^2}$, where d is the number of priorities and m the size of the game.

If we take $n = |\mathcal{A}|$, using the fact that $m = O(2^n)$, we get the complexity $O(2^{4nd+2n})\frac{(2d)!}{d!^2}$, which can be simplified into $O(2^{4n^3+2n}(2n^2)^{n^2}) = O(2^{5n^3+2n})$ using the fact that $d = O(n^2)$. This gives us an exponential bound for the time complexity of this problem. ◀

▶ Remark 38. We can also be interested in the number of tokens needed for Determiniser to witness explorability of an automaton. By inspecting our proof, we can see that we obtain a doubly exponential upper bound. Moreover, we can use the same construction as in [2, Prop 6.3] to show that this is tight, *i.e.* some automata require a doubly exponential number of tokens to witness explorability.

▶ Remark 39. This algorithm only works as such in the case of Büchi automata. The next step would be to adapt it to co-Büchi, with the hope that a solution for both these models might lead to one for parity automata. However, in order to use a similar method in the co-Büchi case, we would want some way to check the winning condition for a play in the explorability game using only the projection of that play in the support arena. This is not possible with the current definitions of these games: we can create plays in the explorability game with the same projection, but different winners. Take the automaton from Figure 2. If we play the 2-explorability game on that automaton, Determiniser has a strategy to ensure that the support are always maximal, alternating between $\{q_0\}$ and $\{q_1, q_2\}$. However, Spoiler can either chose to always take the co-Büchi transition with the same token, or to alternate between tokens. He only wins in the second case.

A.3 ExpTime-hardness of the ω-explorability problem

This part focuses on proving Theorem 22 stating the ExpTime-hardness of the ω-explorability problem for safety automata, which also proves the optimality of the algorithm from Section 4.2.

We reduce from the acceptance problem of a PSPACE alternating Turing machine. This is again inspired from [2].

We take an alternating Turing machine $\mathcal{M} = (\Sigma_\mathcal{M}, Q_\mathcal{M}, \Delta_\mathcal{M}, q_0^\mathcal{M}, q_f^\mathcal{M})$ with $Q_\mathcal{M} = Q_\exists \uplus Q_\forall$. It can be seen as a game between two players: existential ($\exists$) and universal ($\forall$). On a given input, the game creates a run by letting $\exists$ (resp. $\forall$) solve the non-determinism in states from $Q_\exists$ (resp. $Q_\forall$) by picking a transition from Δ. Player $\exists$ wins if the play reaches the accepting state $q_f^\mathcal{M}$, and w is accepted if and only if $\exists$ has a winning strategy. We assume that $\mathcal{M}$ uses polynomial space $P(n)$ in the size n of its input, *i.e.* the winning strategies can avoid configurations with tape longer than $P(n)$. We also fix an input word $w \in (\Sigma_\mathcal{M})^*$.

We will assume for simplicity that $\Sigma_\mathcal{M} = \{0,1\}$ and that the machine alternates between existential and universal states, starting with an existential one (meaning that $q_0 \in Q_\exists$ and the transitions are either $Q_\exists \to Q_\forall$ or $Q_\forall \to Q_\exists$). In our reduction, this will mean that we give the choice of the transition alternatively to Spoiler (playing $\exists$) and Determiniser ($\forall$).

We create a safety automaton $\mathcal{A} = (Q, \Sigma, q_0, \Delta, \bot)$ with:

- $Q = Q_\mathcal{M} \uplus \mathrm{Pos} \uplus \mathrm{Mem} \uplus \mathrm{Trans} \uplus \{q_0, \texttt{store}, \bot, \top\}$ where:

$$\mathrm{Pos} = [1, P(n)]$$
$$\mathrm{Mem} = \{m_{b,i} \mid b \in \{0,1\}, i \in [1, P(n)]\}$$
$$\mathrm{Trans} = \{E\} \cup \{A_t \mid t \in \Delta_\mathcal{M}\}$$

- $\Sigma = \{a_{t,p} \mid t \in \Delta_\mathcal{M} \text{ and } p \in [1, P(n)]\} \uplus \{\texttt{init}, \texttt{end}, \texttt{restart}, \texttt{win}\} \uplus \{\texttt{check}_q \mid q \in Q_\mathcal{M}\} \uplus \{\texttt{check}_{b,i} \mid (b,i) \in \{0,1\} \times [1, P(n)]\}$.
- $\bot$ is a rejecting sink state: a run is accepting if and only if it never reaches this state.

Let us give the intuition for the role of each state of $\mathcal{A}$. First the states in $Q_\mathcal{M}$, Pos and Mem are used to keep track of the configuration of $\mathcal{M}$, as described in Lemma 40. Those in Trans are used to simulate the choices of $\exists$ and $\forall$ (played by Spoiler and Determiniser respectively). The state `store` keeps tokens safe for the remaining of a run when Spoiler decides to ignore their transition choice. The sinks $\top$ and $\bot$ are respectively the one Spoiler must avoid at all cost, and the one in which he wants to send every token eventually.

We now define the transitions in Δ. The states $\top$ and $\bot$ are both sinks ($\top$ accepting and $\bot$ rejecting). We then describe all transitions labelled by the letter $a_{t,p}$ with $p \in \mathrm{Pos}$ and $t = (q, q', b, b', d) \in \Delta_\mathcal{M}$, where q and q' are the starting and destination states of t, while b and b' are the letters read and written at the current head position, and $d \in \{L, R\}$ is the direction taken by the head. These transitions are:

- $q \to q'$.
- $p \to p'$ with $p' = p + 1$ if $d = R$, or $p - 1$ if $d = L$. It goes to $\top$ if $p' \notin [1, P(n)]$.
- $m_{b,p} \to m_{b',p}$, and $m_{b'',p''} \to m_{b'',p''}$ for any b'' and any $p'' \neq p$.
- $E \to A_{t'}$ for any transition t'.
- $A_t \to E$.
- $q'' \to \top$ for any $q'' \neq q$.
- $m_{1-b,p} \to \top$ ($1 - b$ is the boolean negation of b).
- $p' \to \top$ for any $p' \neq p$.
- $A_{t'} \to \texttt{store}$ for any transitions $t' \neq t$.

The first three bullet points manage the evolution of the configuration of $\mathcal{M}$. The next two deal with the alternation between players, and the next three punish Spoiler if the transition is invalid (the `check` letters will handle the case where Determiniser is the one giving an invalid transition). The last one saves the tokens that are not chosen for the transition.

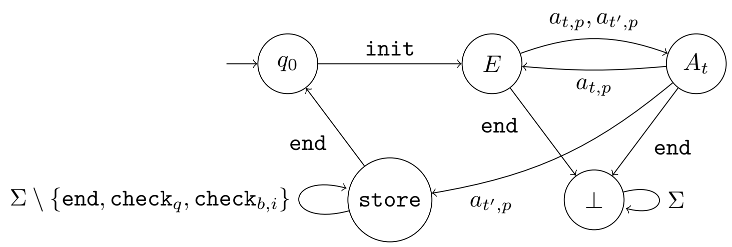

Figure 3 Gadget for simulating the choice of $\forall$ in the alternation (transitions labelled by `check` are not represented, and t' represents any transition different from t).

The other letters give the following transitions.

- `init` goes from q_0 to the states E, $q_0^{\mathcal{M}}$, and $1 \in \text{Pos}$, and also to the states $m_{b,i}$ corresponding to the initial content of the tape, *i.e.* all $m_{b,i}$ such that b is the i-th letter of w (or 0 if $i > |w|$).
- `end` labels transitions from any non accepting state of $\mathcal{M}$ to $\top$, from `store` to q_0, and from any other state to $\bot$.
- $\texttt{check}_q$ creates a transition from A_t to $\bot$ for any $t \in \Delta$ starting from q. It also creates a transition from q to $\top$. Any other state is sent back to q_0. Intuitively, playing that letter means that q is not the current state and that any transition starting from q is invalid.
- $\texttt{check}_{b,i}$ creates a transition from A_t to $\bot$ for any $t \in \Delta$ reading b on the tape. It also creates transitions from any $j \in \text{Pos} \setminus \{i\}$ and from $m_{b,i}$ to $\top$. Any other state is sent to q_0. Intuitively, playing that letter means that the current head position is i, and that its content is not b, so any transition reading b is invalid.

To summarize, the states of $\mathcal{A}$ can be seen as two blocks, apart from q_0, $\top$ and $\bot$: those dealing with the configuration of $\mathcal{M}$ ($Q_{\mathcal{M}}$, Pos and Mem), and those from the gadget of Figure 3 which deal with the alternation and non deterministic choices.

The following result provides tools to manipulate the relation between $\mathcal{A}$ and $\mathcal{M}$.

▶ **Lemma 40.** *Let us consider a play of the ω-explorability game on $\mathcal{A}$, that we stop at some point. Suppose that the letters $a_{t,p}$ played since the last* `init` *are $a_{t_1,p_1}, \ldots, a_{t_k,p_k}$. If $\top$ is not reachable from q_0 with this sequence, then we can define a run ρ of $\mathcal{M}$ on w taking the sequence of transitions $t_1, \ldots, t_k$. The following implications hold:*

Token present in	*implies that at the end of ρ*
$q \in Q_{\mathcal{M}}$	*the current state is q*
$p \in Pos$	*the head is in position p*
$m_{b,i} \in Mem$	*the tape contains b at position i*
E	*it is the turn of $\exists$*
A_t	*it is the turn of $\forall$*

Proof. These results are obtained by straightforward induction from the definitions. The unreachability of $\top$ is used to ensure that only valid transitions are played. ◀

We will now prove that $\mathcal{A}$ is ω-explorable if and only if the Turing machine $\mathcal{M}$ rejects the word w. Let us first assume that $w \in \mathcal{L}(\mathcal{M})$. There is a winning strategy $\sigma_\exists$ for $\exists$ in the alternating Turing machine game, and Spoiler will use that strategy in the explorability game to win against ω tokens. He will consider that the tokens are labelled by integers, and always target the smallest one that is not already in $\bot$. He proceeds as follows.

- Spoiler plays `init` from a position where every token is either in q_0 or $\bot$. We can assume from here that Determiniser sends token to each possible state, and just add imaginary tokens if he does not. Additionally, if the target token does not go to E, then Spoiler creates an imaginary target token in E that will play only valid transitions (we will describe what this means later). Its purpose is to ensure that we actually reach an accepting state of $\mathcal{M}$ to destroy the real target token.
- When there are tokens in E, Spoiler plays letters according to $\sigma_\exists$. More formally, if the letters played since `init` are $a_{t_1,p_1} \dots a_{t_i,p_i}$, then Spoiler plays $a_{t_{i+1},p_{i+1}}$ where $t_{i+1} = \sigma_\exists(t_1, \dots, t_i)$ and $p_{i+1} = p_i + 1$ or $p_i - 1$ depending on the head movement in t_i.
- After such a play, Determiniser can move tokens to any state A_t. If there are more than one occupied state, Spoiler picks the one containing the current target token (possibly imaginary).
  - If that state corresponds to an invalid transition (wrong starting state or wrong tape content at the current head position), then Spoiler plays the corresponding `check` letter. Formally, if the target token (not the imaginary one, since Spoiler can avoid invalid transitions for that one) is in $\mathcal{A}_t$, Spoiler plays $\texttt{check}_q$ if the starting state q of t does not match the current state of the tape (given by Lemma 40), or $\texttt{check}_{b,i}$ if the current head position is i and does not contain b. In both cases, the target token is sent to $\bot$ with no other token reaching $\top$ (by Lemma 40). This sends us back to the first step, but with an updated target.
  - If the state instead corresponds to a valid transition, then Spoiler can play the corresponding $a_{t,p}$, where p is the current head position (again, given by Lemma 40), then go back to the previous step (where there are tokens in E).
- If no invalid transition is reached, the run eventually gets to an accepting state of $\mathcal{M}$ because $\sigma_\exists$ is winning. This corresponds to a stage where Spoiler can safely play `end` to get rid of the target token along with all tokens outside of `store`, by sending them to $\bot$ (the only reason not to play `end` would be the existence of tokens in non accepting states of $Q_\mathcal{M}$). This sends us back to the first step, but with an updated target.

This strategy guarantees that after k runs, at least the first k tokens are in state $\bot$, and therefore cannot witness an accepting run. We also know that the final word is accepted by $\mathcal{A}$, because an accepting run can be created by going to the state `store` as soon as possible in each factor corresponding to a run of $\mathcal{M}$.

Conversely, if there is a winning strategy $\sigma_\forall$ for the universal player in the alternation game on $\mathcal{M}(w)$, then we can build a winning strategy for Determiniser in the ω-explorability game. This strategy is more straightforward than the previous one, as we can focus on the tokens sent to E (while still populating each state when `init` is played, but these other tokens follow a deterministic path until the next `init`).

Determiniser will initially chooses a specific token, called leader. He then sends ω tokens to every reachable state when Spoiler plays `init`, with the leader going to E. Determiniser then moves the tokens in the leader's state according to $\sigma_\forall$. Spoiler cannot send the leader to $\bot$, since the only way to do that would be using the letter `end`, but this would immediately ensure the win for Spoiler, as there will always be some token in non-accepting states of $\mathcal{M}$ (because $\sigma_\forall$ is winning), and those tokens would be sent to $\top$ upon playing `end`. This means that Spoiler has no way to send the leader to $\bot$ without losing the game, and therefore that Determiniser wins.

Note that with that strategy, Spoiler can still safely send some tokens to $\bot$ by playing the wrong transition, which sends the tokens following the leader to `store`, then some well chosen `check` letter to send the remaining ones to $\perp$. However Determiniser will start the next run with still ω tokens, including the leader. This is why the choice of a specific leader is important, as it can never be safely sent to $\perp$.

This proves that the automaton $\mathcal{A}$ created from $\mathcal{M}$ and w (using polynomial time) is ω-explorable if and only if $\mathcal{M}$ rejects w. This completes the proof since the acceptance problem is ExpTime-hard for alternating Turing machines using polynomial space.