Quasi-Weak Cost Automata

A New Variant of Weakness

Denis Kuperberg[1] Michael Vanden Boom[2]

[1]LIAFA/CNRS/Université Paris 7, Denis Diderot, France

[2]Department of Computer Science, University of Oxford, England

FSTTCS 2011
Mumbai

Introduction

- Regular cost functions : counting extension of regular languages

Introduction

- Regular cost functions : counting extension of regular languages
- Motivation : solving bound-related problems on regular languages (e.g. star-height)

Introduction

- Regular cost functions : counting extension of regular languages
- Motivation : solving bound-related problems on regular languages (e.g. star-height)
- Definable over finite or infinite structures, like words or trees

Introduction

- Regular cost functions : counting extension of regular languages
- Motivation : solving bound-related problems on regular languages (e.g. star-height)
- Definable over finite or infinite structures, like words or trees
- Definable via automata, logics, algebraic structures,...

Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$
+ finite set of counters
(initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
(increment `I`, reset `R`, check `C`, no change ε)

Semantics

$[\![\mathcal{A}]\!] : \Sigma^* \to \mathbb{N} \cup \{\infty\}$

Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$
+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment `I`, reset `R`, check `C`, no change ε)

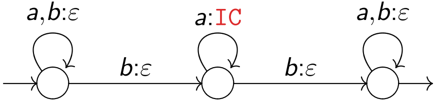
Semantics

$$val_B(\rho) := \text{max checked counter value during run } \rho$$

$$[\![\mathcal{A}]\!]_B(u) := \min\{val_B(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

Example

$[\![\mathcal{A}]\!]_B(u) =$ min length of block of a's surrounded by b's in u



Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$

+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment $\mathtt{I}$, reset $\mathtt{R}$, check $\mathtt{C}$, no change ε)

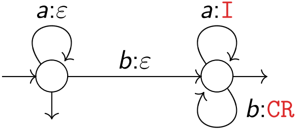
Semantics

$$val_S(\rho) := \text{min checked counter value during run } \rho$$

$$[\![\mathcal{A}]\!]_S(u) := \max\{val_S(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

Example

$[\![\mathcal{A}]\!]_S(u) =$ min length of block of a's surrounded by b's in u



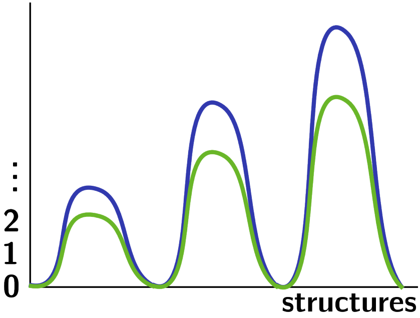
Boundedness relation

"$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$": undecidable [Krob '94]

Boundedness relation

"$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$": undecidable [Krob '94]

"$[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$": decidable on words
[Colcombet '09, following Bojánczyk+Colcombet '06]
for all subsets U, $[\![\mathcal{A}]\!](U)$ bounded iff $[\![\mathcal{B}]\!](U)$ bounded

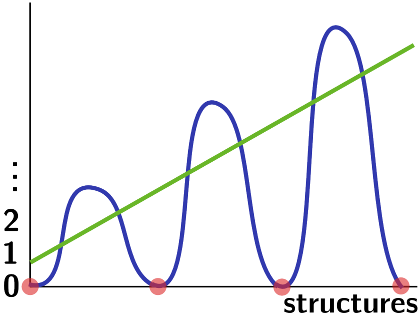


$[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$

Boundedness relation

"$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$": undecidable [Krob '94]

"$[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$": decidable on words
[Colcombet '09, following Bojánczyk+Colcombet '06]
for all subsets U, $[\![\mathcal{A}]\!](U)$ bounded iff $[\![\mathcal{B}]\!](U)$ bounded



$[\![\mathcal{A}]\!] \not\approx [\![\mathcal{B}]\!]$

Applications

Many problems for a regular language L can be reduced to deciding $\approx$ for some class of automata with counting features:

- **Finite power property** (finite words)
  [Simon '78, Hashiguchi '79]

  is there some n such that $(L + \epsilon)^n = L^*$?

- **Star-height problem** (finite words/trees)
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]

  given n, is there a regular expression for L
  with at most n nestings of Kleene star?

- **Parity-index problem** (infinite trees)
  [reduction in Colcombet+Löding '08, decidability open]

  given $i < j$, is there a parity automaton for L
  which uses only priorities $\{i, i+1, \ldots, j\}$?

Applications

Many problems for a regular language L can be reduced to deciding $\approx$ for some class of automata with counting features:

- **Finite power property** (finite words)
  [Simon '78, Hashiguchi '79] — distance

  is there some n such that $(L + \epsilon)^n = L^*$?

- **Star-height problem** (finite words/trees)
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08] — nested distance-desert

  given n, is there a regular expression for L with at most n nestings of Kleene star?

- **Parity-index problem** (infinite trees)
  [reduction in Colcombet+Löding '08, decidability open] — cost-parity

  given $i < j$, is there a parity automaton for L which uses only priorities $\{i, i+1, \ldots, j\}$?

Cost functions on infinite trees

- In the following, input structures = Σ-labelled infinite trees.

Cost functions on infinite trees

- In the following, input structures = Σ-labelled infinite trees.
- Dual B- and S- semantics as before, defining functions : Trees$\to \mathbb{N} \cup \{\infty\}$.

Cost functions on infinite trees

- In the following, input structures = Σ-labelled infinite trees.
- Dual B- and S- semantics as before, defining functions : Trees$\rightarrow \mathbb{N} \cup \{\infty\}$.
- Acceptance condition : any condition on infinite words : Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).

Cost functions on infinite trees

- In the following, input structures = Σ-labelled infinite trees.
- Dual B- and S- semantics as before, defining functions : Trees$\to \mathbb{N} \cup \{\infty\}$.
- Acceptance condition : any condition on infinite words : Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).
- Decidability of $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$ open in general.

Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language L can be viewed as a B- or S-automaton without any counters. Then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language L can be viewed as a B- or S-automaton without any counters. Then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

- Switching between B and S semantics corresponds to a complementation.

Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language L can be viewed as a B- or S-automaton without any counters. Then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

- Switching between B and S semantics corresponds to a complementation.
- If L and L' are languages, $\chi_L \approx \chi_{L'}$ iff $L = L'$, so cost function theory, even up to $\approx$, strictly extends language theory.

Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language L can be viewed as a B- or S-automaton without any counters. Then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

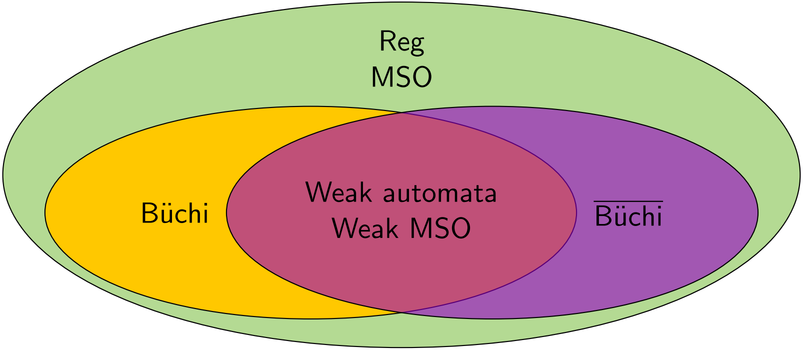
- Switching between B and S semantics corresponds to a complementation.
- If L and L' are languages, $\chi_L \approx \chi_{L'}$ iff $L = L'$, so cost function theory, even up to $\approx$, strictly extends language theory.
- Aim : Extend classic theorems from languages to cost functions

Rabin-style characterization

Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language L of infinite trees is recognizable by
an alternating weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

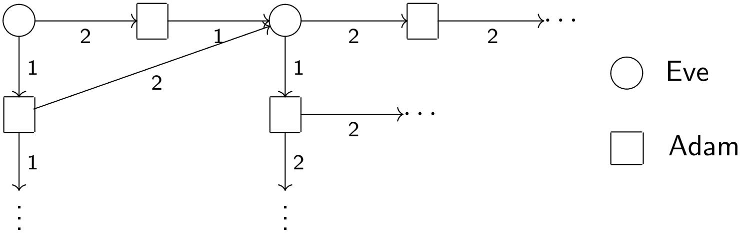
$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$



Weak automata and games

Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
$\Rightarrow \exists M. \forall t.$ any play of $(\mathcal{A}, t)$ has at most M alternations between priorities

Game $(\mathcal{A}, t)$



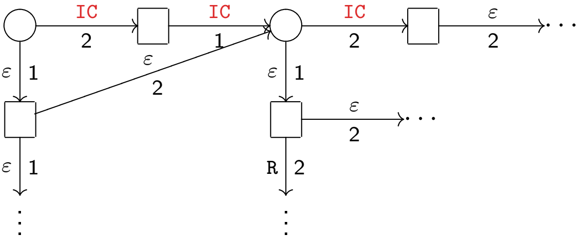
Semantics

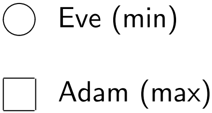
A strategy σ for Eve is winning if every play in σ stabilizes in priority 2
$\mathcal{A}$ accepts t if Eve has a winning strategy from the initial position

Weak B-automata and games [Vanden Boom '11]

Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
$\Rightarrow \exists M.\forall t.$ any play of $(\mathcal{A}, t)$ has at most M alternations between priorities
+ finite set of counters and counter actions $\mathtt{I}, \mathtt{R}, \mathtt{C}, \varepsilon$ on transitions

Game $(\mathcal{A}, t)$





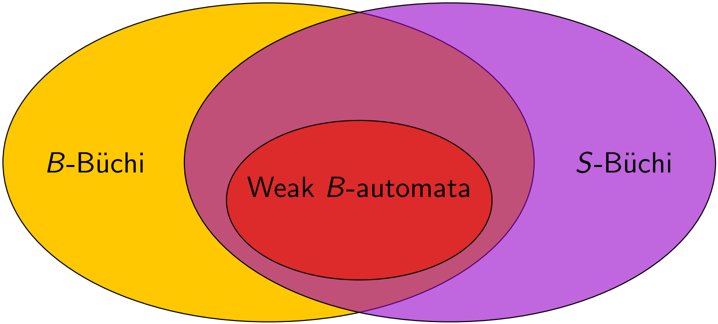
Semantics

$val(\sigma) :=$ max value of any play in strategy σ

$[\![\mathcal{A}]\!](t) := \min\{val(\sigma) : \sigma$ is a winning strategy for Eve in $(\mathcal{A}, t)\}$

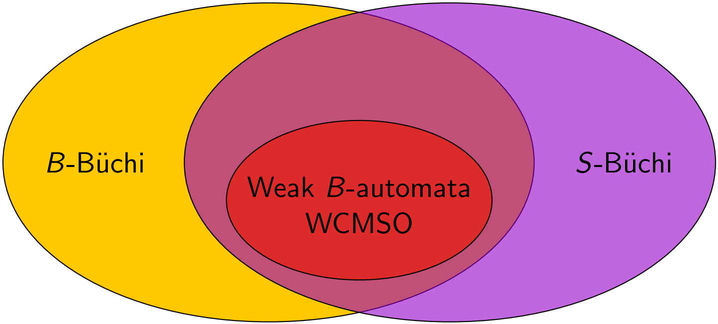
Results on weak cost functions [Vanden Boom '11]

- Translation from weak to nondeterminist B-Büchi, S-Büchi



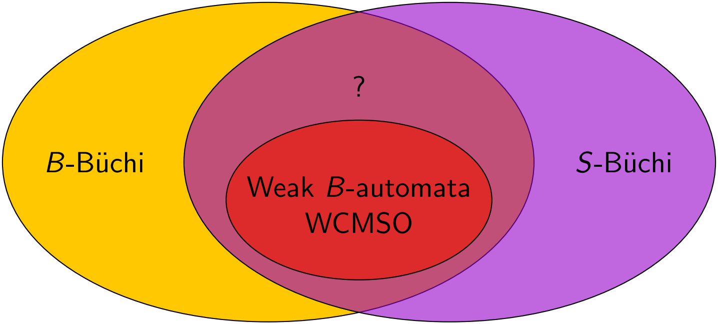
Results on weak cost functions [Vanden Boom '11]

- Translation from weak to nondeterminist B-Büchi, S-Büchi
- Good closure properties of the weak class, equivalence with logic.



Results on weak cost functions [Vanden Boom '11]

- Translation from weak to nondeterminist B-Büchi, S-Büchi
- Good closure properties of the weak class, equivalence with logic.
- Does Rabin theorem extend to the weak cost function class ?



Rabin-style characterization

Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language L of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$

Rabin-style characterization

Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language L of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$

Conjecture

A cost function f on infinite trees is recognizable by
a **weak** B**-automaton** iff
there exists a nondeterministic B-Büchi automaton $\mathcal{U}$ and
a nondeterministic S-Büchi automaton $\mathcal{U}'$ such that

$$f \approx [\![\mathcal{U}]\!]_B \approx [\![\mathcal{U}']\!]_S.$$

Rabin-style characterization

Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language L of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$

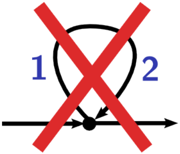
Theorem

A cost function f on infinite trees is recognizable by
a **quasi-weak** B**-automaton**
iff there exists a nondeterministic B-Büchi automaton $\mathcal{U}$ and
a nondeterministic S-Büchi automaton $\mathcal{U}'$ such that

$$f \approx [\![\mathcal{U}]\!]_B \approx [\![\mathcal{U}']\!]_S.$$

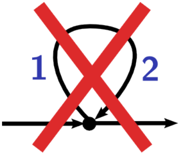
Variants of weakness

Weak B-automaton	**Quasi-weak B-automaton**
alternating B-Büchi	
$\exists M.\forall t.$ any play in $(\mathcal{A}, t)$ has at most M alternations between priorities	
there is no cycle with both priorities	



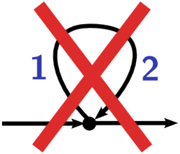
Variants of weakness

Weak B-automaton	Quasi-weak B-automaton
alternating B-Büchi	alternating B-Büchi
$\exists M. \forall t.$ any play in $(\mathcal{A}, t)$ has at most M alternations between priorities	
there is no cycle with both priorities	



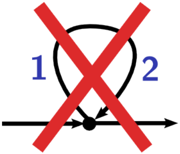
Variants of weakness

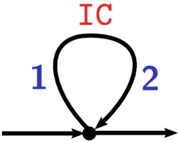
Weak B-automaton	**Quasi-weak B-automaton**
alternating B-Büchi	alternating B-Büchi
$\exists M. \forall t.$ any play in $(\mathcal{A}, t)$ has at most M alternations between priorities	$\forall N. \exists M. \forall t. \forall \sigma$ for Eve in $(\mathcal{A}, t)$. $val(\sigma) \leq N \rightarrow$ any play in σ has at most M alternations between priorities
there is no cycle with both priorities	



Variants of weakness

Weak B-automaton	Quasi-weak B-automaton
alternating B-Büchi	alternating B-Büchi
$\exists M.\forall t.$ any play in $(\mathcal{A}, t)$ has at most M alternations between priorities	$\forall N.\exists M.\forall t.\forall \sigma$ for Eve in $(\mathcal{A}, t)$. $val(\sigma) \leq N \rightarrow$ any play in σ has at most M alternations between priorities
there is no cycle with both priorities	if there is a cycle with both priorities, then there is some IC without R

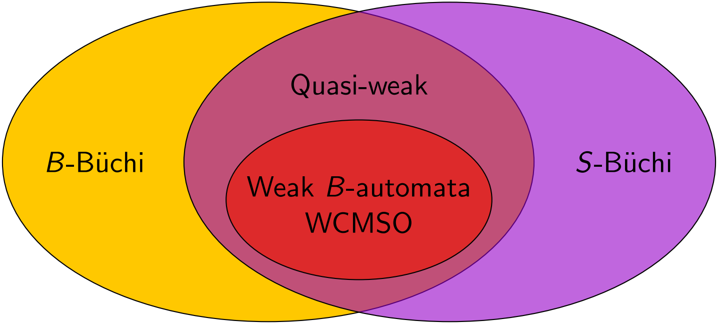




Separation Result

Theorem

There is a quasi-weak cost function which is not weak.



Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99

Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-weak B-automata, then it is **decidable** whether or not $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.

Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-weak B-automata, then it is **decidable** whether or not $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.
- Quasi-weak B-automata are **strictly more expressive** than weak B-automata over infinite trees.

Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-weak B-automata, then it is **decidable** whether or not $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.
- Quasi-weak B-automata are **strictly more expressive** than weak B-automata over infinite trees.

Quasi-weak B-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-weak B-automata, then it is **decidable** whether or not $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.
- Quasi-weak B-automata are **strictly more expressive** than weak B-automata over infinite trees.

Quasi-weak B-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Corollary : If $\mathcal{A}$ is a Büchi automaton, it is decidable whether $L(\mathcal{A})$ is a weak language [using Colcombet+Löding 08]

Summary and conclusion

Theorem

- Quasi-weak B-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak B-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi '99
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-weak B-automata, then it is **decidable** whether or not $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.
- Quasi-weak B-automata are **strictly more expressive** than weak B-automata over infinite trees.

Quasi-weak B-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Corollary : If $\mathcal{A}$ is a Büchi automaton, it is decidable whether $L(\mathcal{A})$ is a weak language [using Colcombet+Löding 08]

Is $\approx$ decidable for **cost-parity automata**?