Quasi-Weak Cost Automata
A New Variant of Weakness

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FSTTCS 2011
Mumbai
Introduction

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- Motivation: solving bound-related problems on regular languages (e.g. star-height)
- Definable over finite or infinite structures, like words or trees
- Definable via automata, logics, algebraic structures,...
Cost automata over words

Nondeterministic finite-state automaton \( \mathcal{A} \)
+ finite set of counters
  (initialized to 0, values range over \( \mathbb{N} \))
+ counter operations on transitions
  (increment \( I \), reset \( R \), check \( C \), no change \( \varepsilon \))

**Semantics**

\[
[\mathcal{A}] : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}
\]
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Semantics

\[ \text{val}_B(\rho) := \max \text{ checked counter value during run } \rho \]

\[ [\mathcal{A}]_B(u) := \min \{\text{val}_B(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\} \]

Example

\[ [\mathcal{A}]_B(u) = \min \text{ length of block of } a \text{'s surrounded by } b \text{'s in } u \]
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\[ \text{val}_{S}(\rho) := \min \text{ checked counter value during run } \rho \]
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Boundedness relation

“$[A] = [B]$”: undecidable [Krob '94]

$[A](U)$ bounded iff $[B](U)$ bounded
Boundedness relation

“\([\mathcal{A}] = [\mathcal{B}]\)” : undecidable [Krob '94]

“\([\mathcal{A}] \approx [\mathcal{B}]\)” : decidable on words
[Colcombet '09, following Bojánczyk+Colcombet '06]
for all subsets \(U\), \([\mathcal{A}] (U)\) bounded iff \([\mathcal{B}] (U)\) bounded
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for all subsets $U$, $[\mathcal{A}](U)$ bounded iff $[\mathcal{B}](U)$ bounded
Many problems for a regular language $L$ can be reduced to deciding $\approx$ for some class of automata with counting features:

- **Finite power property** (finite words)
  [Simon ’78, Hashiguchi ’79]
  is there some $n$ such that $(L + \epsilon)^n = L^*$?

- **Star-height problem** (finite words/trees)
  [Hashiguchi ’88, Kirsten ’05, Colcombet+Löding ’08]
  given $n$, is there a regular expression for $L$ with at most $n$ nestings of Kleene star?

- **Parity-index problem** (infinite trees)
  [reduction in Colcombet+Löding ’08, decidability open]
  given $i < j$, is there a parity automaton for $L$ which uses only priorities $\{i, i + 1, \ldots, j\}$?
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Acceptance condition : any condition on infinite words : Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).
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- Dual $B$- and $S$- semantics as before, defining functions: $\text{Trees} \to \mathbb{N} \cup \{\infty\}$.
- Acceptance condition: any condition on infinite words: Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).
- Decidability of $[A] \approx [B]$ open in general.
Languages as cost functions

A standard automaton $\mathcal{A}$ computing a language $L$ can be viewed as a $B$- or $S$-automaton without any counters. Then $\llbracket \mathcal{A} \rrbracket_B = \chi_L$ and $\llbracket \mathcal{A} \rrbracket_S = \chi_L^\complement$, with

$$
\chi_L(t) = \begin{cases} 
0 & \text{if } t \in L \\
\infty & \text{if } t \notin L 
\end{cases}
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Switching between $B$ and $S$ semantics corresponds to a complementation.
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- If $L$ and $L'$ are languages, $\chi_L \approx \chi_L'$ iff $L = L'$, so cost function theory, even up to $\approx$, strictly extends language theory.
A standard automaton $A$ computing a language $L$ can be viewed as a $B$- or $S$-automaton without any counters. Then $[A]_B = \chi_L$ and $[A]_S = \chi_L$, with

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Aim: Extend classic theorems from languages to cost functions
Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language $L$ of infinite trees is recognizable by an alternating weak automaton iff there are nondeterministic Büchi automata $U$ and $U'$ such that

$$L = L(U) = \overline{L(U')}.$$
Weak automata and games

Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$

+ no cycle in the transition function which visits both priorities

$\Rightarrow \exists M. \forall t. \text{ any play of } (\mathcal{A}, t) \text{ has at most } M \text{ alternations between priorities}$

**Game** $(\mathcal{A}, t)$

![Game Diagram]

**Semantics**

A strategy $\sigma$ for Eve is winning if every play in $\sigma$ stabilizes in priority 2

$\mathcal{A}$ accepts $t$ if Eve has a winning strategy from the initial position
Alternating parity automaton $A$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
$\Rightarrow \exists M. \forall t. \text{any play of } (A, t) \text{ has at most } M \text{ alternations between priorities}$
+ finite set of counters and counter actions $I, R, C, \epsilon$ on transitions

**Game** $(A, t)$

```
Game (A, t)
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```
Semantics

$\text{val}(\sigma) := \max \text{ value of any play in strategy } \sigma$

$\llbracket A \rrbracket(t) := \min \{\text{val}(\sigma) : \sigma \text{ is a winning strategy for Eve in } (A, t)\}$
```
Results on weak cost functions [Vanden Boom ’11]

- Translation from weak to nondeterminist $B$-Büchi, $S$-Büchi

![Venn diagram showing the overlap between $B$-Büchi, $S$-Büchi, and Weak $B$-automata sets.](attachment:venn_diagram.png)
Translation from weak to nondeterministic $B$-Büchi, $S$-Büchi

Good closure properties of the weak class, equivalence with logic.
Translation from weak to nondeterminist $B$-Büchi, $S$-Büchi

Good closure properties of the weak class, equivalence with logic.

Does Rabin theorem extend to the weak cost function class?
Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language $L$ of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$
Rabin-style characterization

**Theorem (Rabin 1970, Kupferman + Vardi 1999)**

A language \( L \) of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata \( U \) and \( U' \) such that

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L = L(U) = \overline{L(U')}.
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**Conjecture**

A cost function \( f \) on infinite trees is recognizable by a weak \( B \)-automaton iff there exists a nondeterministic \( B \)-Büchi automaton \( U \) and a nondeterministic \( S \)-Büchi automaton \( U' \) such that

\[
f \approx [U]_B \approx [U']_S.
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**Theorem (Rabin 1970, Kupferman + Vardi 1999)**

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**Theorem**

A cost function $f$ on infinite trees is recognizable by a **quasi-weak $B$-automaton**

iff there exists a nondeterministic $B$-Büchi automaton $U$ and a nondeterministic $S$-Büchi automaton $U'$ such that

$$f \approx [U]_B \approx [U']_S .$$
## Variants of weakness

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<th>Quasi-weak $B$-automaton</th>
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There is no cyclical transition between priorities 1 and 2.
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![Diagram](image)
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![Diagram showing a cycle with priorities 1 and 2](image-url)
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Theorem

There is a quasi-weak cost function which is not weak.
Summary and conclusion

Theorem

- Quasi-weak $B$-automata have a **Rabin-style characterization**.
  - Quasi-weak $\rightarrow$ nondeterministic: relies on the fact that finite-memory strategies suffice in quasi-weak $B$-games
  - Nondeterministic $\rightarrow$ quasi-weak: adapt Kupferman+Vardi ’99

If $A$ and $B$ are quasi-weak $B$-automata, then it is decidable whether or not $[A] \approx [B]$.

Quasi-weak $B$-automata are strictly more expressive than weak $B$-automata over infinite trees. Quasi-weak $B$-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Corollary: If $A$ is a Büchi automaton, it is decidable whether $L(A)$ is a weak language [using Colcombet+Löding 08]

Is $\approx$ decidable for cost-parity automata?
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Is $\approx$ decidable for **cost-parity automata**?