The theory of regular cost functions, from finite words to infinite trees.

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Introduction

- **Church-Turing 1936:**
  - Which problems can be answered by an algorithm?
  - It has yield the notion of decidability.
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- **Automata theory:**
  Toolbox to decide many problems arising naturally.
  Verification of systems can be done automatically.
  Theoretical and practical advantages.
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- **Automata theory:** Toolbox to decide many problems arising naturally. Verification of systems can be done automatically. Theoretical and practical advantages.

- **Problem:** Decidability is still open for some automata-related problems.
1 Automata theory

2 Regular Cost Functions

3 Contributions of the thesis

4 Zoom: Aperiodic Cost Functions
Finite Automaton

A word $u \in \Sigma^*$ is **accepted** by $\mathcal{A}$ if there is an accepting path labeled by $u$.

**Example**: Accepting path for the word $babc$. 

![Finite Automaton Diagram]
Finite Automaton

A word $u \in A^*$ is accepted by $A$ if there is an accepting path labeled by $u$.

Example: Accepting path for the word $babc$. 
Finite Automaton

A word $u \in \Delta^*$ is accepted by $\mathcal{A}$ if there is an accepting path labeled by $u$:

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Example: Accepting path for the word $babc$. 

\[
\begin{array}{c}
q_0 \overset{b, c}{\rightarrow} q_0 \overset{a}{\rightarrow} q_1 \overset{c}{\rightarrow} q_1 \overset{a}{\rightarrow} q_2 \overset{b}{\rightarrow} q_2
\end{array}
\]
A word $u \in \mathbb{A}^*$ is accepted by $A$ if there is an accepting path labeled by $u$:

**Example**: Accepting path for the word $babc$. 

A finite automaton is a mathematical model of computation that accepts or rejects strings of symbols and produces a unique computation of whether the string is accepted or rejected. Finite automata are widely used in the field of computer science, particularly in the study of formal languages and compilers.
Finite Automaton

A word $u \in \mathbb{A}^*$ is **accepted** by $\mathcal{A}$ if there is an accepting path labeled by $u$:

**Example**: Accepting path for the word $babc$.

The **language recognized** by $\mathcal{A}$ is the set $L \subseteq \mathbb{A}^*$ of words accepted by $\mathcal{A}$. 

\[
\begin{array}{c}
q_0 \xrightarrow{a} q_1 \\
q_0 \xrightarrow{b, c} q_0 \\
q_0 \xrightarrow{c} q_1 \\
q_1 \xrightarrow{a} q_1 \\
q_1 \xrightarrow{b} q_2 \\
q_2 \xrightarrow{a, b, c} q_2 \\
q_2 \xrightarrow{b} q_2 \\
q_2 \xrightarrow{c} q_2
\end{array}
\]
Descriptions of a language

Language recognized: \( L_{ab} = \{ \text{words containing } ab \} \).

Other ways than automata to specify \( L_{ab} \):

- Regular expression: \( A^* ab A^* \),
**Descriptions of a language**

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Other ways than automata to specify \( L_{ab} \):

- **Regular expression**: \( A^* ab A^* \),

- **Logical sentence (MSO)**: \( \exists x \ \exists y \ a(x) \land b(y) \land (y = Sx) \).
Descriptions of a language

Language recognized: \( L_{ab} = \{ \text{words containing } ab \} \).

Other ways than automata to specify \( L_{ab} \):

- Regular expression: \( A^* ab A^* \),
- Logical sentence (MSO): \( \exists x \ \exists y \ a(x) \land b(y) \land (y = Sx) \).
- Finite monoid: \( M = \{ 1, a, b, c, ba, 0 \}, \ P = \{ 0 \} \)
  \( ab = 0, \ aa = ca = a, \ bb = bc = b, \ cc = ac = cb = c \)
All these formalisms are effectively equivalent.
All these formalisms are effectively equivalent.

\[ a^n b^n \]

**Regular Languages**
- Expressions
- MSO
- Monoids
- Automata

**Star-free Languages**
- Star-free Expressions
- FO
- Aperiodic Monoids
- Counter-free Automata
Historical motivation

Given a class of languages $C$, is there an algorithm which given an automaton for $L$, decides whether $L \in C$?

**Theorem (Schützenberger 1965)**

*It is decidable whether a regular language is star-free, thanks to the equivalence with aperiodic monoids.*
Historical motivation

Given a class of languages $C$, is there an algorithm which given an automaton for $L$, decides whether $L \in C$?

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**Finite Power Problem:** Given $L$, is there $n$ such that

$$(L + \varepsilon)^n = L^*$$

There is no known algebraic characterization, other technics are needed to show decidability.
Distance Automata

$A_1$: number of $a$

Unbounded: There are words with arbitrarily large value.

Deciding **Boundedness** for distance automata $\Rightarrow$ solving finite power problem.

**Theorem (Hashiguchi 82, Kirsten 05)**

*Boundedness is decidable for distance automata.*
Problems solved using counters

- **Finite Power** (finite words) [Simon ’78, Hashiguchi ’79]
  Is there $n$ such that $(L + \varepsilon)^n = L^*$?

- **Fixed Point Iteration** (finite words)
  [Blumensath+Otto+Weyer ’09]
  Can we bound the number of fixpoint iterations in a MSO formula?

- **Star-Height** (finite words/trees)
  [Hashiguchi ’88, Kirsten ’05, Colcombet+Löding ’08]
  Given $n$, is there an expression for $L$, with at most $n$ nesting of Kleene stars?

- **Parity Rank** (infinite trees)
  [reduction in Colcombet+Löding ’08, decidability open, deterministic input Niwinski+Walukiewicz ’05]
  Given $i < j$, is there a parity automaton for $L$ using ranks $\{i, i + 1, \ldots, j\}$?
1. Automata theory

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Theory of Regular Cost Functions

**Aim:** General framework for previous constructions.

- Generalize from languages $L : \mathbb{A}^* \rightarrow \{0, 1\}$
  to functions $f : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{\infty\}$

- Accordingly generalize automata, logics, semigroups, in order to obtain a theory of regular cost functions, which behaves as well as possible.

- Obtain decidability results thanks to this new theory.
Cost automata over words

Nondeterministic finite-state automaton $A$
+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment $I$, reset $R$, check $C$, no change $\varepsilon$)

Semantics: $[A] : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$
Cost automata over words

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**Semantics:** $[[A]] : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$

$\text{val}_B(\rho) := \text{max checked counter value during run } \rho$

$[[A]]_B(u) := \min\{\text{val}_B(\rho) : \rho \text{ is an accepting run of } A \text{ on } u\}$

**Example**

$[[A]]_B(u) = \min \text{ length of block of } a's \text{ surrounded by } b's \text{ in } u$

```
  a, b: ε  a: IC  a, b: ε
  b: ε    b: ε    b: ε
```

Diagram of cost automaton:
Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$
+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment $I$, reset $R$, check $C$, no change $\varepsilon$)

**Semantics:**

$$[\mathcal{A}] : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$$

$val_S(\rho) := \min$ checked counter value during run $\rho$

$$[\mathcal{A}]_S(u) := \max\{val_S(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

**Example**

$$[\mathcal{A}]_S(u) = \min \text{ length of block of } a's \text{ surrounded by } b's \text{ in } u$$

\begin{center}
\begin{tikzpicture}
  \node[state] (a) at (-1,0) {};
  \node[state] (b) at (1,0) {};

  \draw[-stealth] (a) edge node[above]{$a: \varepsilon$} (b);
  \draw[-stealth] (b) edge node[below]{$b: \varepsilon$} (a);
  \draw[-stealth] (a) edge node[above]{$a: I$} (b);
  \draw[-stealth] (b) edge node[below]{$b: CR$} (a);
\end{tikzpicture}
\end{center}
Boundedness relation

“$[A] = [B]$”: undecidable [Krob ’94]
Boundedness relation

“\([A] = [B]\)”: undecidable [Krob ’94]

“\([A] \approx [B]\)”: decidable on words

[Colcombet ’09, following Bojánczyk+Colcombet ’06]

for all subsets \(U\), \([A](U)\) bounded iff \([B](U)\) bounded
Boundedness relation

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for all subsets $U$, $[A](U)$ bounded iff $[B](U)$ bounded

\[ [A] \not\approx [B] \]
Therefore we always identify two functions if they are bounded on the same sets.

**Example**

For any function \( f \), we have \( f \approx 2f \approx \exp(f) \).

But \((u \mapsto |u|_a) \not\approx (u \mapsto |u|_b)\), as witnessed by the set \( a^* \).
Therefore we always identify two functions if they are bounded on the same sets.

**Example**

For any function $f$, we have $f \approx 2f \approx \exp(f)$.

But $(u \mapsto |u|_a) \not\approx (u \mapsto |u|_b)$, as witnessed by the set $a^*$. 

**Theorem (Colcombet ’09, following Hashiguchi, Leung, Simon, Kirsten, Bojańczyk+Colcombet)**

Cost automata $\Leftrightarrow$ Cost logics $\Leftrightarrow$ Stabilisation monoids.

For some suitable models of Cost Logics and Stabilisation Monoids, extending the classical ones.

Boundedness decidable.

All these equivalences are only valid up to $\approx$.

It provides a toolbox to decide boundedness problems.
Languages as cost functions

A language $L$ is represented by its characteristic function:

$$\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$$

If $A$ is a classical automaton for $L$, then $[A]_B = \chi_L$ and $[A]_S = \chi_L$. Switching between $B$ and $S$ is the generalization of language complementation.

Cost function theory strictly extends language theory.

All theorems on cost functions are in particular true for languages.

**Goal of the thesis:** Studying cost function theory, and generalise known theorems from languages to cost functions.
1. Automata theory

2. Regular Cost Functions

3. Contributions of the thesis

4. Zoom: Aperiodic Cost Functions
Contributions of the thesis

Input structures:

Finite words: \textit{accba}

Infinite words: \textit{abaabaccbaba} \ldots

Infinite trees:

\begin{tikzpicture}
  \node (a) {a}
    child {node (b) {b}
      child {node (c) {c}}
    }
    child {node (c) {c}
      child {node (a) {a}}
    }
    child {node (b) {b}
      child {node (a) {a}}
    }
  \end{tikzpicture} \quad \ldots
Contributions of the thesis

Input structures:

Finite words: accba

Infinite words: abaabaccbaba ...

Infinite trees: a  b  c  a  c
                b  c  a
                c  b  b
           a  c  b

Different kinds of results:

- Generalisation of language notions and theorems,
- Study of classes specific to cost functions,
- Reduction of classical decision problems to boundedness problems.
Cost Functions on finite words

Decidability of membership and effectiveness of translations
[Colcombet+K.+Lombardy ICALP '10, K. STACS '11].
Generalization of Myhill-Nerode Equivalence [K. STACS '11].
Boundedness of CLTL is PSPACE-complete [Submitted to LMCS].
Cost Functions on infinite words

Regular Functions
- CMSO
- B/S-Büchi automata
- WCMSO
- Weak B-automata

Aperiodic Functions
- Very-Weak Automata
- CFO
- CLTL

Decidability of membership and effectiveness of translations
[K.+Vanden Boom, ICALP ’12].
Languages on infinite trees

**Theorem (Rabin 1970, Kupferman + Vardi 1999)**

$L$ recognizable by an alternating weak automaton ⇔
$L$ recognizable by WMSO ⇔ there are Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that $L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}$. 

![Diagram showing the relationship between different automata theories](image-url)
Cost functions on infinite trees

- Weak $B$-automata
  - WCMSO


If $A$ is a Büchi automaton, it is decidable whether $L(A)$ is weak [submitted to LICS '13].

Logic for the Quasi-Weak class [Submitted to ICALP '13].
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Cost Functions on finite words
Logics on Finite Words

- **First-Order Logic (FO):** we quantify over positions in the word.

\[ \varphi := a(x) \mid x \leq y \mid \neg \varphi \mid \varphi \lor \psi \mid \exists x \varphi \]
Logics on Finite Words

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  \[ \varphi := a(x) \mid x \leq y \mid \neg \varphi \mid \varphi \lor \psi \mid \exists x \varphi \]

- **MSO:** FO with quantification on sets, noted \( X, Y \).
Logics on Finite Words

- **First-Order Logic (FO)**: we quantify over positions in the word.

\[ \varphi := a(x) \mid x \leq y \mid \neg \varphi \mid \varphi \lor \psi \mid \exists x \varphi \]

- **MSO**: FO with quantification on sets, noted \( X \), \( Y \).

- **Linear Temporal Logic (LTL) over \( \mathbb{A}^* \)**:

\[ \varphi := a \mid \Omega \mid \neg \varphi \mid \varphi \lor \psi \mid X\varphi \mid \varphi U \psi \]

\[ \varphi U \psi : \quad a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8 \ a_9 \ a_{10} \]

Future operators **G** (Always) and **F** (Eventually).

**Example**: To describe \( L_{ab} \), we can write \( F(a \land Xb) \).
Generalisation: cost LTL

CLTL over $A^*$:

$$\varphi ::= a \mid \Omega \mid \varphi \land \psi \mid \varphi \lor \psi \mid X\varphi \mid \varphi U\psi \mid \varphi U^{\leq N}\psi$$

Negations pushed to the leaves.
Generalisation: cost LTL

- **CLTL** over $A^*$:

  $$\varphi := a \mid \Omega \mid \varphi \land \psi \mid \varphi \lor \psi \mid X\varphi \mid \varphi U\psi \mid \varphi U^{\leq N}\psi$$

  Negations pushed to the leaves.

- $\varphi U^{\leq N}\psi$ means that $\psi$ is true in the future, and $\varphi$ is false at most $N$ times in the mean time.

  $$\varphi U^{\leq N}\psi: \quad \varphi \varphi \times \varphi \varphi \times \varphi \varphi \psi$$

  $$a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}$$
Generalisation: cost LTL

- **CLTL** over $\mathbb{A}^*$:

$$\varphi := a \mid \Omega \mid \varphi \land \psi \mid \varphi \lor \psi \mid X\varphi \mid \varphi U\psi \mid \varphi U^{\leq N} \psi$$

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- “Error variable” $N$ is unique, shared by all occurrences of $U^{\leq N}$. 
Generalisation: cost LTL

- **CLTL** over $A^*$:

  \[ \varphi := a \mid \Omega \mid \varphi \land \psi \mid \varphi \lor \psi \mid X\varphi \mid \varphi U \psi \mid \varphi U \leq N \psi \]

  Negations pushed to the leaves.

- $\varphi U \leq N \psi$ means that $\psi$ is true in the future, and $\varphi$ is false at most $N$ times in the mean time.

  \[ \varphi U \leq N \psi: \quad \varphi \varphi \times \varphi \varphi \times \varphi \varphi \psi \]
  \[ \quad a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} \]

- “Error variable” $N$ is unique, shared by all occurrences of $U \leq N$.

- $G \leq N \varphi$: $\varphi$ is false at most $N$ times in the future ($\varphi U \leq N \Omega$).
Generalisation: Cost FO and Cost MSO

- **CFO** over $A^*$:

  \[ \varphi := a(x) \mid x = y \mid x < y \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \exists x \varphi \mid \forall x \varphi \mid \forall \leq N x \varphi \]

  Negations pushed to the leaves.

- As before, $N$ unique free variable.

- $\forall \leq N x \varphi(x)$ means $\varphi$ is false on at most $N$ positions.

- **CMSO** extends CFO by allowing quantification over sets.
From formula to cost function:
Formula $\varphi \rightarrow$ cost function $[\varphi] : A^* \rightarrow \mathbb{N} \cup \{\infty\}$, defined by

$$[\varphi](u) = \inf\{n \in \mathbb{N} : \varphi \text{ is true over } u \text{ with } n \text{ as error value}\}$$

Example with the alphabet $\{a, b\}$

- $\text{number}_a = [G^{\leq_N} b] = [\forall^{\leq_N} x \ b(x)]$. 
Semantics of Cost Logics

From formula to cost function:
Formula $\varphi \rightarrow$ cost function $[\varphi] : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{\infty\}$, defined by

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Example with the alphabet $\{a, b\}$

- $\text{number}_a = [G \leq_N b] = [\forall \leq_N x \ b(x)]$.
- $\text{maxblock}_a = [G(\perp U \leq_N (b \lor \Omega))]$
  $$= [\forall X \ \text{block}_a(X) \Rightarrow (\forall \leq_N x \ x \notin X)].$$
From formula to cost function:

Formula $\varphi \rightarrow$ cost function $[\varphi] : A^* \rightarrow \mathbb{N} \cup \{\infty\}$, defined by

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Example with the alphabet $\{a, b\}$

- number$_a = [G \leq^N b] = [\forall \leq^N x \ b(x)].$
- maxblock$_a = [G(\bot U \leq^N (b \lor \Omega))]$
  $$= [\forall X \ \text{block}_a(X) \Rightarrow (\forall \leq^N x \ x \notin X)].$$
- If $\varphi$ is a classical formula for $L$, then $[\varphi] = \chi_L$. 
Stabilisation monoids

- **Aim:** Generalise monoids to a quantitative setting.
Stabilisation monoids

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- if we “count” \(a\), then \(a^# \neq a\), otherwise \(a^# = a\).
**Aim:** Generalise monoids to a quantitative setting.

Stabilisation $\#$ means “repeat many times” the element.

if we “count” $a$, then $a^\# \neq a$, otherwise $a^\# = a$.

**Example:** Stabilisation Monoid for number $a$

$M = \{b, a, 0\}$, $P = \{a, b\}$,

$b$: “no $a$”, $a$: “a little number of $a$”, $0$: “a lot of $a$”.

---

![Cayley graph](image-url)
**Definition:** A [stabilisation] monoid $M$ is *aperiodic* if for all $x \in M$ there is $n \in \mathbb{N}$ such that $x^n = x^{n+1}$. 
**Aperiodic Monoids**

**Definition:** A [stabilisation] monoid $M$ is *aperiodic* if for all $x \in M$ there is $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

**Theorem (McNaughton-Papert, Schützenberger, Kamp)**

$\text{Aperiodic Monoids} \iff \text{FO} \iff \text{LTL} \iff \text{Star-free Expressions.}$

We want to generalise this theorem to cost functions. The problems are:

- No complementation $\Rightarrow$ No Star-free expressions.
- Deterministic automata are strictly weaker.
- Heavy formalisms (semantics of stabilisation monoids).
- New quantitative behaviours.
- Original proofs already hard.
Aperiodic cost functions

Theorem (K. STACS 2011)

Aperiodic stabilisation monoid $\Leftrightarrow$ CLTL $\Leftrightarrow$ CFO.

Proof Ideas:

- Generalisation of Myhill-Nerode $\Rightarrow$ Syntactic object.
- Induction on $(|M|, |A|)$.
- Extend functions to sequences of words.
- Use bounded approximations.
- Extend CLTL with Past operators, show Separability.
Thank you!