Quasi-Weak Cost Functions
A New Variant of Weakness

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Introduction

- Regular cost functions: counting extension of regular languages
- Motivation: solving bound-related problems on regular languages (e.g. star-height)
- Definable over finite or infinite structures, like words or trees
- Definable via automata, logics, algebraic structures,...
Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$
+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment $I$, reset $R$, check $C$, no change $\varepsilon$)

**Semantics**

$[\mathcal{A}] : A^* \rightarrow \mathbb{N} \cup \{\infty\}$
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Semantics

$val_B(\rho) := \max$ checked counter value during run $\rho$

$[A]_B(u) := \min \{ val_B(\rho) : \rho$ is an accepting run of $A$ on $u \}$

Example

$[A]_B(u) = \min$ length of block of $a$’s surrounded by $b$’s in $u$
Cost automata over words

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**Semantics**

\begin{align*}
  \text{val}_S(\rho) & := \text{min checked counter value during run } \rho \\
  [\mathcal{A}]_S(u) & := \max\{\text{val}_S(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}
\end{align*}

**Example**

$[\mathcal{A}]_S(u) = \text{min length of block of } a\text{'s surrounded by } b\text{'s in } u$
Boundedness relation

“$[A] = [B]$”: undecidable [Krob ’94]
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“$[A] \approx [B]$”: decidable on words

[Colcombet ’09, following Bojánczyk+Colcombet ’06]
for all subsets $U$, $[A](U)$ bounded iff $[B](U)$ bounded
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"$[\mathcal{A}] \approx [\mathcal{B}]$": decidable on words
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for all subsets $U$, $[\mathcal{A}](U)$ bounded iff $[\mathcal{B}](U)$ bounded
In the following, input structures = $A$-labelled infinite trees.
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Acceptance condition: any condition on infinite words: Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).
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Decidability of $[\mathcal{A}] \approx [\mathcal{B}]$ open in general.
A standard automaton \(\mathcal{A}\) computing a language \(L\) can be viewed as a \(B\)- or \(S\)-automaton without any counters. Then \([\mathcal{A}]_B = \chi_L\) and \([\mathcal{A}]_S = \chi_{\overline{L}}\), with

\[
\chi_L(t) = \begin{cases} 
0 & \text{if } t \in L \\
\infty & \text{if } t \notin L
\end{cases}
\]
Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language $L$ can be viewed as a $B$- or $S$-automaton without any counters. Then $[\mathcal{A}]_B = \chi_L$ and $[\mathcal{A}]_S = \chi_L$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

- Switching between $B$ and $S$ semantics corresponds to a complementation.
Languages as cost functions

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- Switching between $B$ and $S$ semantics corresponds to a complementation.

- If $L$ and $L'$ are languages, $\chi_L \approx \chi_{L'}$ iff $L = L'$, so cost function theory, even up to $\approx$, strictly extends language theory.
Languages as cost functions

A standard automaton \( A \) computing a language \( L \) can be viewed as a \( B \)- or \( S \)-automaton without any counters. Then \( \llbracket A \rrbracket_B = \chi_L \) and \( \llbracket A \rrbracket_S = \chi_L^\bot \), with

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Switching between \( B \) and \( S \) semantics corresponds to a complementation.

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Aim: Extend classic theorems from languages to cost functions
Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language $L$ of infinite trees is recognizable by an alternating weak automaton iff there are nondeterministic Büchi automata $U$ and $U'$ such that

$$L = L(U) = \overline{L(U')}.$$
Weak automata and games

Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
⇒ $\exists M. \forall t$. any play of $(\mathcal{A}, t)$ has at most $M$ alternations between priorities

Game $(\mathcal{A}, t)$

Semantics
A strategy $\sigma$ for Eve is winning if every play in $\sigma$ stabilizes in priority 2
$\mathcal{A}$ accepts $t$ if Eve has a winning strategy from the initial position
Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
$\Rightarrow \exists M. \forall t. \text{ any play of } (\mathcal{A}, t) \text{ has at most } M \text{ alternations }$ between priorities
+ finite set of counters and counter actions $I, R, C, \varepsilon$ on transitions

### Game $(\mathcal{A}, t)$

![Game Diagram]

- Eve (min)
- Adam (max)

### Semantics

$val(\sigma) := \text{max value of any play in strategy } \sigma$

$[\mathcal{A}](t) := \text{min}\{val(\sigma) : \sigma \text{ is a winning strategy for Eve in } (\mathcal{A}, t)\}$
Translation from weak to nondeterminist $B$-Büchi, $S$-Büchi
Results on weak cost functions [Vanden Boom ’11]

- Translation from weak to nondeterminist $B$-Büchi, $S$-Büchi
- Good closure properties of the weak class, equivalence with logic.
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- Translation from weak to nondeterminist $B$-Büchi, $S$-Büchi
- Good closure properties of the weak class, equivalence with logic.
- Does Rabin theorem extend to the weak cost function class?
Rabin-style characterization

**Theorem (Rabin 1970, Kupferman + Vardi 1999)**

A language $L$ of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata $U$ and $U'$ such that

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**Conjecture**

A cost function $f$ on infinite trees is recognizable by a **weak $B$-automaton** iff there exists a nondeterministic $B$-Büchi automaton $U$ and a nondeterministic $S$-Büchi automaton $U'$ such that

$$f \approx [U]_B \approx [U']_S.$$
Rabin-style characterization

**Theorem (Rabin 1970, Kupferman + Vardi 1999)**

A language \( L \) of infinite trees is recognizable by a weak automaton iff there are nondeterministic Büchi automata \( U \) and \( U' \) such that

\[
L = L(U) = \overline{L(U')}.
\]

**Theorem (KV ’11)**

A cost function \( f \) on infinite trees is recognizable by a **quasi-weak \( B \)-automaton**

iff there exists a nondeterministic \( B \)-Büchi automaton \( U \) and a nondeterministic \( S \)-Büchi automaton \( U' \) such that

\[
f \approx \llbracket U \rrbracket_B \approx \llbracket U' \rrbracket_S.
\]
# Variants of weakness

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- $A$ is a set of states, $t$ a term, $M$ an integer.
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![Diagram of alternating $B$- Büchi automaton with a cycle between priorities 1 and 2 marked as forbidden](image)
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![Diagram](image)
Theorem (KV '11)

There is a quasi-weak cost function which is not weak.
Logical equivalents

- **Monadic Second-Order logic**:
  - **Weak MSO**: Set variables range over *finite* sets.
  - **CWMSO [Vanden Boom '11]**: Weak MSO + $|X| \leq N$. 
    \[
    \mathcal{L}(t) = \inf\{n| \varphi[N \leftarrow n] \text{ is true} \}.
    \]
  - **QWMSO [BCKPV '14]**: WCMSO + $\mu^N x. \varphi(x)$. 
    Fixpoint operator with a bounded number of expansions.
Logical equivalents

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  - **QWMSO** [BCKPV ’14]: WCMSO + \(\mu^N x. \varphi(x)\).
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- **\(\mu\)-calculus**
  - **alternation free**: no \(\nu\)-scope intersects a \(\mu\)-scope

- **\(\mu^N\)-calculus** [BCKPV ’14]: \(\mu\)-calculus + \(\mu^N x. \varphi(x)\).
  - **Weak**: no \(\nu\)-scope intersects a \(\mu\)-scope, and no \(\mu^N\)-scope simultaneously intersects a \(\mu\)-scope and a \(\nu\)-scope.
  - **Quasi-Weak** if no \(\nu\)-scope intersects a \(\mu\)-scope.

Proofs use two-way alternating \(B\)-automata as an intermediary tool.
Open question: Given a regular language $L$, is it weak?

Partial Answer [CKLV '13]: If $L$ is Büchi then we can decide it.

Construction:

1. Start from a nondeterministic Büchi automaton $U$ for $L$; dual $U$ is coBüchi for $L$.
2. Build automaton $W = U_{\text{Acc}} \cup U_{\text{Rej}}$, with $U_{\text{Acc}}$ Büchi $\rightarrow U_{\text{Rej}}$ and $U_{\text{Rej}}$ Eve: IC $\rightarrow U_{\text{Acc}}$.
3. $W$ is Quasi-weak, and we can show that $[W] \approx \chi_L$ iff $L$ is weak.

- If $[W] \approx \chi_L$, then $L$ is weak by just storing counter values of $W$ up to the bound.
- If $t \in L$, then $[W](t) = \infty$ since Adam can play an accepting run of $U$. 
- If $L$ is weak, Kupferman-Vardi construction $\Rightarrow \sigma_{\text{Eve}}$ bounded when $t /\in L$. 


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Construction:

- Start from a nondeterministic Büchi automaton \( \mathcal{U} \) for \( L \), dual \( \overline{\mathcal{U}} \) is coBüchi for \( \overline{L} \).
- Build automaton \( \mathcal{W} = \overline{U}_{\text{Acc}} \cup \overline{U}_{\text{Rej}} \),
  with \( \overline{U}_{\text{Acc}} \xrightarrow{\text{Büchi}} \overline{U}_{\text{Rej}} \) and \( \overline{U}_{\text{Rej}} \xrightarrow{\text{Eve: IC}} \overline{U}_{\text{Acc}} \).

\[ \overline{W} \approx \chi_L \text{ iff } \mathcal{W} \]
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- If $L$ is weak, Kupferman-Vardi construction $\Rightarrow \sigma_{Eve}$ bounded when $t \notin L$. 
Summary and conclusion

Theorem

- Quasi-weak $B$-automata have characterizations in term of
  - Büchi cost functions (Rabin-style)
  - Cost MSO
  - $\mu^N$-calculus

If $A$ and $B$ are Quasi-weak $B$-automata, then it is decidable whether or not $[A] \approx [B]$.

Quasi-weak $B$-automata are strictly more expressive than weak $B$-automata over infinite trees.

If $A$ is a Büchi automaton, it is decidable whether $L(A)$ is a weak language.

Quasi-weak $B$-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Is $\approx$ decidable for cost-parity automata?
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- If $\mathcal{A}$ and $\mathcal{B}$ are Quasi-weak $B$-automata, then it is **decidable** whether or not $[[\mathcal{A}]] \approx [[\mathcal{B}]]$.

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