Quasi-Weak Cost Functions

A New Variant of Weakness

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Introduction

- Regular cost functions : counting extension of regular languages
- Motivation : solving bound-related problems on regular languages (e.g. star-height)
- Definable over finite or infinite structures, like words or trees
- Definable via automata, logics, algebraic structures,...

Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$
+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions
  (increment `I`, reset `R`, check `C`, no change ε)

Semantics

$[\![\mathcal{A}]\!] : A^* \to \mathbb{N} \cup \{\infty\}$

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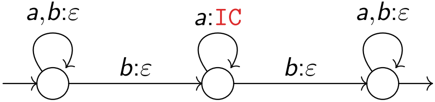
Semantics

$$val_B(\rho) := \text{max checked counter value during run } \rho$$

$$[\![\mathcal{A}]\!]_B(u) := \min\{val_B(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

Example

$[\![\mathcal{A}]\!]_B(u) =$ min length of block of a's surrounded by b's in u



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$$val_S(\rho) := \text{min checked counter value during run } \rho$$
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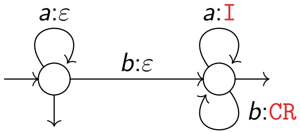

Boundedness relation

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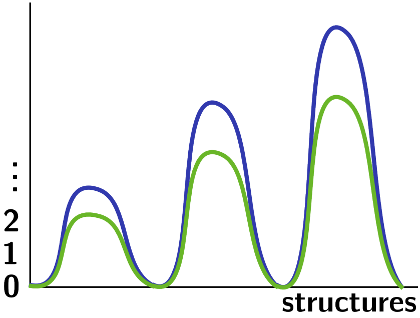
[Colcombet '09, following Bojánczyk+Colcombet '06]
for all subsets U, $[\![\mathcal{A}]\!](U)$ bounded iff $[\![\mathcal{B}]\!](U)$ bounded



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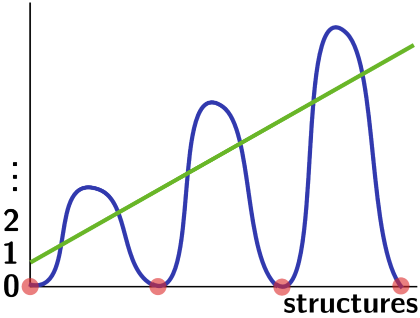

$[\![\mathcal{A}]\!] \not\approx [\![\mathcal{B}]\!]$

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- Acceptance condition : any condition on infinite words : Büchi, co-Büchi, Rabin, Parity,... (on all branches in the non-deterministic setting).
- Decidability of $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$ open in general.

Languages as cost functions

- A standard automaton $\mathcal{A}$ computing a language L can be viewed as a B- or S-automaton without any counters. Then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$, with

$$\chi_L(t) = \begin{cases} 0 & \text{if } t \in L \\ \infty & \text{if } t \notin L \end{cases}$$

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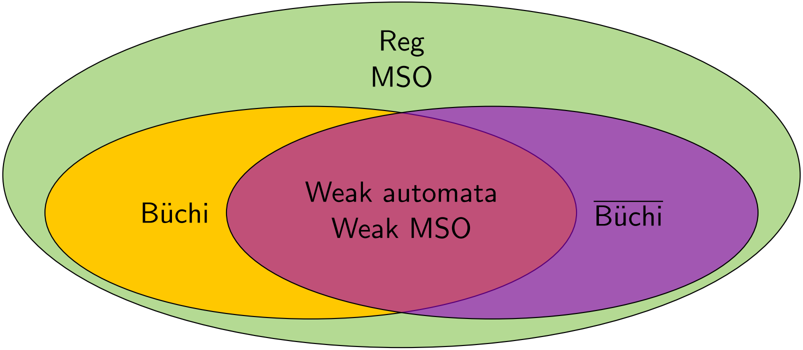
- Aim : Extend classic theorems from languages to cost functions

Rabin-style characterization

Theorem (Rabin 1970, Kupferman + Vardi 1999)

A language L of infinite trees is recognizable by
an alternating weak automaton iff there are nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that

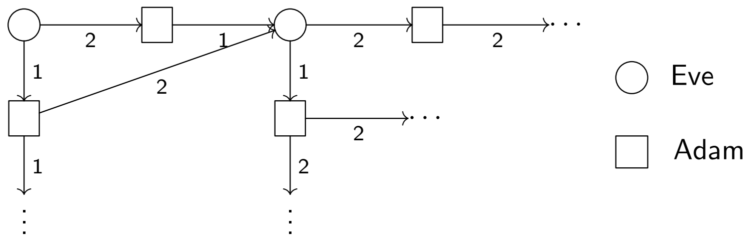
$$L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}.$$



Weak automata and games

Alternating parity automaton $\mathcal{A}$ with priorities $\{1, 2\}$
+ no cycle in the transition function which visits both priorities
$\Rightarrow \exists M. \forall t.$ any play of $(\mathcal{A}, t)$ has at most M alternations between priorities

Game $(\mathcal{A}, t)$



Semantics

A strategy σ for Eve is winning if every play in σ stabilizes in priority 2
$\mathcal{A}$ accepts t if Eve has a winning strategy from the initial position

Weak B-automata and games [Vanden Boom '11]

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$\Rightarrow \exists M. \forall t.$ any play of $(\mathcal{A}, t)$ has at most M alternations between priorities
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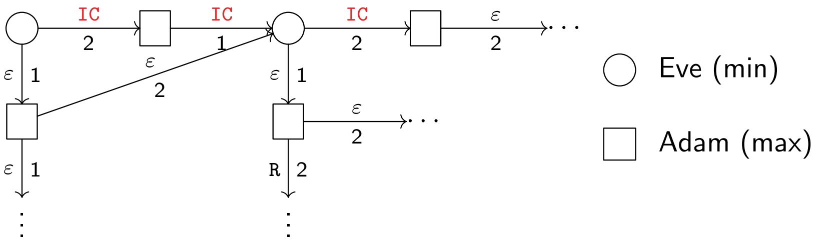

Semantics

$$val(\sigma) := \text{max value of any play in strategy } \sigma$$

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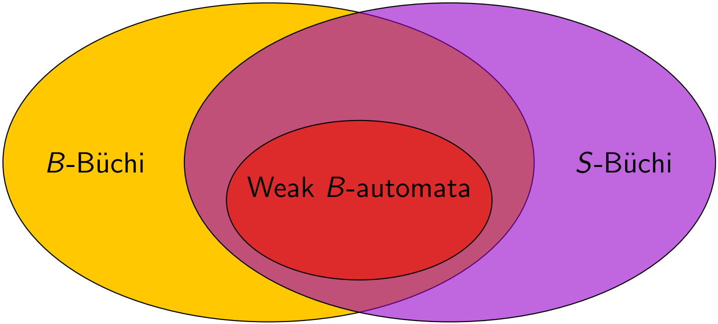
Results on weak cost functions [Vanden Boom '11]

- Translation from weak to nondeterminist B-Büchi, S-Büchi



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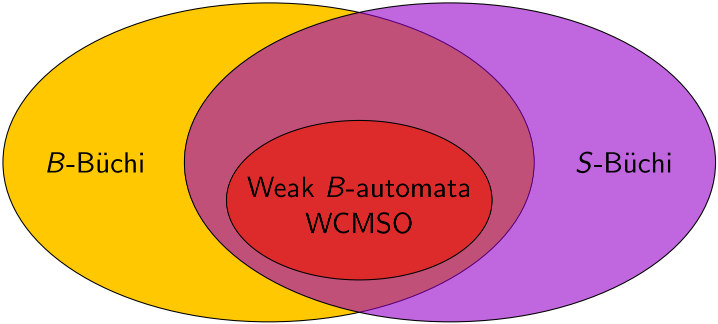

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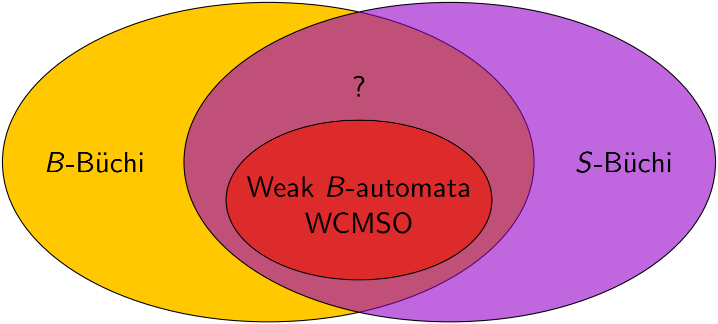
- Does Rabin theorem extend to the weak cost function class ?



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Conjecture

A cost function f on infinite trees is recognizable by
a **weak** B-**automaton** iff
there exists a nondeterministic B-Büchi automaton $\mathcal{U}$ and
a nondeterministic S-Büchi automaton $\mathcal{U}'$ such that

$$f \approx [\![\mathcal{U}]\!]_B \approx [\![\mathcal{U}']\!]_S.$$

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Theorem (KV '11)

A cost function f on infinite trees is recognizable by
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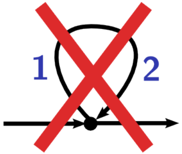
Variants of weakness

Weak B-automaton	Quasi-weak B-automaton
alternating B-Büchi	
$\exists M . \forall t$. any play in $(\mathcal{A}, t)$ has at most M alternations between priorities	
there is no cycle with both priorities	



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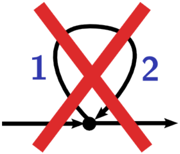

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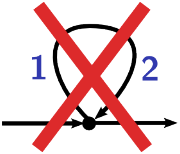

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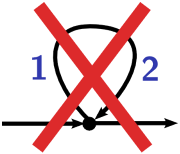
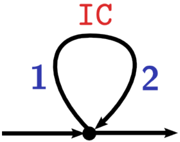
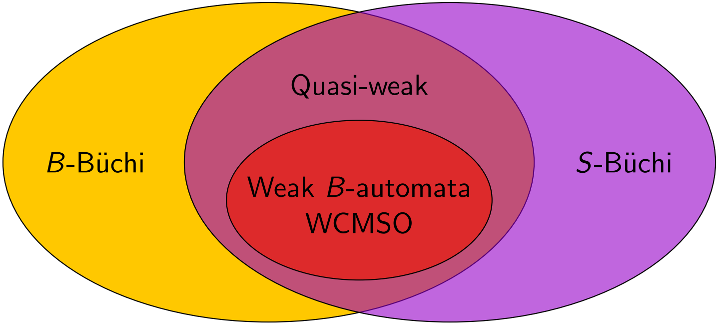
there is no cycle with both priorities	if there is a cycle with both priorities, then there is some IC without R
	

Separation Result

Theorem (KV '11)

There is a quasi-weak cost function which is not weak.



Logical equivalents

- Monadic Second-Order logic :
  - Weak MSO : Set variables range over *finite* sets.
  - CWMSO [Vanden Boom '11] : Weak MSO + $|X| \leq N$.
    $[\![\varphi]\!](t) = \inf\{n | \varphi[N \leftarrow n] \text{ is true }\}$.
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- μ-calculus
  - alternation free : no ν-scope intersects a μ-scope
  - μ^N-calculus [BCKPV '14] : μ-calculus + $\mu^N x.\varphi(x)$.
    - **Weak** : no ν-scope intersects a μ-scope, and no μ^N-scope simultaneously intersects a μ-scope and a ν-scope.
    - **Quasi-Weak** if no ν-scope intersects a μ-scope.

Proofs use two-way alternating B-automata as an intermediary tool.

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Construction :

- Start from a nondeterministic Büchi automaton $\mathcal{U}$ for L, dual $\overline{\mathcal{U}}$ is coBüchi for $\overline{L}$.
- Build automaton $\mathcal{W} = \overline{\mathcal{U}}_{Acc} \cup \overline{\mathcal{U}}_{Rej}$, with $\overline{\mathcal{U}}_{Acc} \xrightarrow{\text{Büchi}} \overline{\mathcal{U}}_{Rej}$ and $\overline{\mathcal{U}}_{Rej} \xrightarrow{\text{Eve:}\texttt{IC}} \overline{\mathcal{U}}_{Acc}$.

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- If $[\![\mathcal{W}]\!] \approx \chi_{\overline{L}}$, then $\overline{L}$ is weak by just storing counter values of $\mathcal{W}$ up to the bound.
- If $t \in L$, then $[\![\mathcal{W}]\!](t) = \infty$ since Adam can play an accepting run of $\mathcal{U}$
- If L is weak, Kupferman-Vardi construction $\Rightarrow \sigma_{Eve}$ bounded when $t \notin L$.

Summary and conclusion

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- If $\mathcal{A}$ is a Büchi automaton, it is decidable whether $L(\mathcal{A})$ is a weak language

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Quasi-weak B-automata extend the class of cost automata over infinite trees for which $\approx$ is known to be decidable.

Is $\approx$ decidable for **cost-parity automata**?