Comment compter avec LTL

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Introduction

- Regular cost functions : counting extension of regular languages

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- $LTL^{\leq}$: new simple way to define regular cost functions
- Translation from $LTL^{\leq}$ to automata
- Algebraic characterization of $LTL^{\leq}$-definable cost functions

Outline

Introduction

Counting events in words
- B-automata
- Cost functions

Quantitative Linear temporal logic
- Definition
- Semantics

Algebraic characterization
- Stabilization monoids

B-automata

Aim : To represent functions $\mathbb{A}^* \longrightarrow \mathbb{N} \cup \{\infty\}$ with automata.

B-automaton :

- nondeterministic finite-state
- finite set of counters, ranging over $\mathbb{N}$, initial value 0
- each transition performs actions on each counter

Atomic actions : increment (i), reset (r), do nothing (ε).

Semantics of B-automata

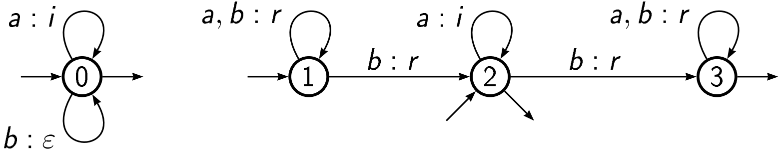
Function $\mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$ associated with a B-automaton $\mathcal{A}$:

$[\![\mathcal{A}]\!]_B(u) = \inf\{n/$ there is a run where all counter values stay below $n\}$

with $\inf \emptyset = \infty$.

Example

$[\![\mathcal{A}]\!]_B = |\cdot|_a$ *and* $[\![\mathcal{A}']\!]_B : a^{n_1}ba^{n_2}\cdots ba^{n_k} \mapsto \min(n_1, n_2, \ldots, n_k)$



More on B-automata

Remark : A standard automaton $\mathcal{A}$ computing L can be viewed as a B-automaton without any counter.
Then $[\![\mathcal{A}]\!]_B = \chi_L$ with $\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$

Theorem ([Krob '94])

The equivalence of two distance automata (particular case of B-automata) is undecidable.

How to get a decidable quantitative extension of regular languages ?
Solution : Loosing some precision on the counting, but keeping information about bounds.

Cost functions

If $f, g : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, then

$$f \approx g \text{ if } \forall X \subseteq \mathbb{A}^*, f|_X \text{ bounded } \Leftrightarrow g|_X \text{ bounded.}$$

Cost function : equivalence class for $\approx$ relation.

Example

For $\mathbb{A} = \{a, b, c\}$,

- $\max(|\cdot|_a, |\cdot|_b) \approx |\cdot|_a + |\cdot|_b$,
- $|\cdot|_a \not\approx \mathrm{maxblock}_a$ *: on* $X = (ab)^*$*, only* $\mathrm{maxblock}_a$ *is bounded.*

Known results on B-automata

Extension of the notion of language via $\chi_L : L = L' \Leftrightarrow \chi_L \approx \chi_{L'}$.

Theorem (Colcombet '09)

It is decidable whether two B-automata compute the same cost function (modulo $\approx$).

B-automata-computable cost functions are called regular.

Example

χ_L *is a regular cost function iff* L *is a regular language.*

- LTL on $\mathbb{A}$ describes regular languages :
  $\varphi := a \mid \Omega \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid X\varphi \mid \varphi U \psi$
  where Ω marks the end of the word.

$$\varphi U \psi : \quad \begin{matrix} \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \psi & & \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{matrix}$$

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- $\varphi U^{\leq N} \psi$ means that ψ is true somewhere in the future, and φ is false at most N times until then.

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- The "error value" variable N is unique, and is shared by all occurrences of $U^{\leq N}$ operator.

Semantics of LTL$^{\leq}$

From formula to cost function :

$[\![\varphi]\!]$ is the cost function associated to φ, defined by

$$[\![\varphi]\!](u) = \inf\{N \in \mathbb{N}, \varphi \text{ is true on } u \text{ with N as error value}\}$$

Example

For all $u \in \{a, b\}^*$, we have

- $[\![bU^{\leq N}\Omega]\!](u) = |u|_a$.
- $[\![G(\bot U^{\leq N}(b \vee \Omega))]\!](a^{n_0}ba^{n_1}b \cdots ba^{n_k}) = \max(n_0, n_1, \ldots, n_k)$
- $[\![F(b \wedge X(\bot U^{\leq N}(b \vee \Omega)))]\!](a^{n_0}ba^{n_1}b \cdots ba^{n_k}) = \min(n_1, \ldots, n_k)$

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Theorem

We can effectively translate an LTL$^{\leq}$-formula φ into a B-automaton with $2^{|\varphi|}$ states.

Reminder :

- regular language $\Leftrightarrow$ finite monoid (Myhill)
- LTL$\Leftrightarrow$ star-free $\Leftrightarrow$ aperiodic monoid (Schützenberger)
- Syntactic congruence for L : $u \sim_L v$ if $\forall x, y \in \mathbb{A}^*, xuy \in L \Leftrightarrow xvy \in L$

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Stabilization monoid : $\mathbf{S} = \langle S, \cdot, \leq, \sharp \rangle$, ordered monoid with a $\sharp$-operator : stabilization over idempotents ($e = e \cdot e$).

$e^\sharp$ means "e repeated a lot of times".

Syntactic congruence for cost functions

- ♯-expressions : words enriched with exponent ♯,
- Context $C[]$: ♯-expression with a hole.
- Syntactic congruence for f : over ♯-expressions u, v

$$u \sim_f v \text{ if } \forall C[], f(C[u](n)) \to \infty \Leftrightarrow f(C[v](n)) \to \infty$$

where $u(n)$ is the word obtained by replacing ♯ with n in u

Quotienting ♯-expressions by $\sim_f$ $\longrightarrow$ minimal stabilization monoid for f

Algebraic characterization of regular cost functions

Regular languages	Regular cost functions
Classic automaton	B-automaton [Colcombet '09]
LTL	$\text{LTL}^{\leq}$
Finite monoid	Finite stabilization monoid [Col09]
Minimal monoid	Minimal stabilization monoid [Colcombet, K., Lombardy '10]
Syntactic congruence	Cost functions syntactic congruence
LTL $\Leftrightarrow$ aperiodic monoid [Kamp,McNaughton&Papert, Schützenberger]	$\text{LTL}^{\leq}$ $\Leftrightarrow$ aperiodic stabilization monoid [K. '11]

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Corollary

The class of $\mathrm{LTL}^{\leq}$-definable cost functions is decidable.

Conclusion

Summary

- Definition of $LTL^{\leq}$ to easily describe cost functions
- Translation from $LTL^{\leq}$ to B-automata
- Syntactic congruence for cost functions
- Algebraic characterization and decidability of the class of $LTL^{\leq}$-definable cost functions.

Future work

- Extension to infinite words
- Other characterizations of this class by first-order logic, star-free expressions,. . .