Linear temporal logic for regular cost functions

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Introduction

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- $\mathrm{LTL}^{\leq}$: simple way to define cost functions

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- $LTL^{\leq}$: simple way to define cost functions
- Translation from $LTL^{\leq}$ to automata
- Algebraic characterization of $LTL^{\leq}$-definable cost functions

Outline

B-automata

Aim : To represent functions $\mathbb{A}^* \longrightarrow \mathbb{N} \cup \{\infty\}$ with automata.

B-automaton :

- nondeterministic finite-state
- finite set of counters, ranging over $\mathbb{N}$, initial value 0
- each transition perform actions on each counter

Atomic actions : increment (i), reset (r), check (c).

Semantics of B-automata

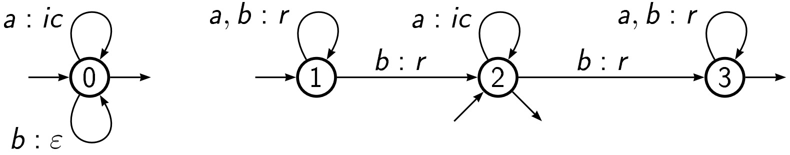
Cost function associated with a B-automton $\mathcal{A}$:

$$[\![\mathcal{A}]\!]_B(u) = \inf\{n /\ \text{there is a run with maximal check value } n\}$$

with $\inf \emptyset = \infty$.

Example

$[\![\mathcal{A}]\!]_B = |\cdot|_a$ *and* $[\![\mathcal{A}']\!]_B = \mathrm{minblock}_a : u \mapsto \min\{n / a^n \textit{ factor of } u\}$



More on B-automata

Remark : A standard automaton $\mathcal{A}$ computing L can be viewed as a B-automaton without any counter.

Then $[\![\mathcal{A}]\!]_B = \chi_L$ with $\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$

Theorem ([Krob 94])

The equivalence of two B-automata is undecidable.

Solution : Loosing some precision on the counting, but keeping information about bounds.

Cost functions

If $f, g : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, then

$$f \approx g \text{ if } \forall X \subseteq \mathbb{A}^*, f|_X \text{ bounded } \Leftrightarrow g|_X \text{ bounded.}$$

iff $\exists\, \alpha : \mathbb{N} \to \mathbb{N}$ such as $f \leq \alpha \circ g$ and $g \leq \alpha \circ f$ (with $\alpha(\infty) = \infty$)

Cost function : equivalence class for $\approx$ relation.

Example

For $\mathbb{A} = \{a, b, c\}$,

$\max(|\cdot|_a, |\cdot|_b) \ \approx \ |\cdot|_a + |\cdot|_b$ *but* $|\cdot|_a \ \not\approx \ \mathrm{maxblock}_a$

Known results on B-automata

Extension of the notion of language via $\chi_L : L \neq L' \implies \chi_L \not\approx \chi_{L'}$.

Theorem (Colcombet 09)

It is decidable whether two B-automata compute the same cost function (modulo $\approx$).

Automata-computable cost functions are called regular.

Example

χ_L *is a regular cost function iff L is a regular language.*

- LTL on $\mathbb{A}$ describes regular languages :
  $\varphi := a \mid \Omega \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X\varphi \mid \varphi U \varphi$
  where Ω marks the end of the word.

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- $\varphi U^{\leq N} \psi$ means that ψ is true somewhere in the future, and φ is false at most N times until then.
- The variable N is unique, and is shared by all occurrences of $U^{\leq N}$ operator.

Semantics of $LTL^{\leq}$

Satisfiability with integer parameter :

We write $(u, n) \models \varphi$ to signify that $u \in \mathbb{A}^*$ satisfies the formula φ, with $n \in \mathbb{N}$ as value for all the occurences of N in φ.

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$(aabbcbccacacb, 5) \models (aU^{\leq N}b)U\Omega$

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From formula to cost function :

$[\![\varphi]\!]$ is the cost function associated to φ, defined by

$$[\![\varphi]\!](u) = \inf\{n \in \mathbb{N}, (u, n) \models \varphi\}$$

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For all $u \in \{a, b\}^*$*, we have* $[\![bU^{\leq N}\Omega]\!](u) = |u|_a$.

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Theorem

We can effectively translate an LTL$^{\leq}$*-formula* φ *into a B-automaton with* $2^{|\varphi|}$ *states.*

Algebraic characterization of regular cost functions

Reminder :

- regular language $\Leftrightarrow$ finite semigroup (Myhill)
- LTL$\Leftrightarrow$ star-free $\Leftrightarrow$ aperiodic semigroup (Schützenberger)
- Nerode equivalence for L : $u \sim_L v$ if
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Stabilization semigroup : $\mathbf{S} = \langle S, \cdot, \leq, \sharp \rangle$, ordered semigroup with a $\sharp$-operator : stabilization over idempotents ($e = e \cdot e$).

$e^\sharp$ means "e repeated a lot of times".

Theorem (Colcombet 09)

Stabilization semigroups recognize exactly the set of regular cost functions, and translations to or from B-automata are effective.

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Theorem

Aperiodic stabilization semigroups recognize exactly the set of $LTL^{\leq}$-definable cost functions, hence this class is decidable.

Conclusion

Summary

- Definition of LTL$^{\leq}$ to easily describe cost functions
- Translation from LTL$^{\leq}$ to B-automata
- Syntactic congruence for cost functions
- Algebraic characterization and decidability of the class of LTL$^{\leq}$-definable cost functions.

Future work

- Extension to infinite words, trees,...
- Other characterizations of this class by first-order logic, star-free expressions,...