The theory of regular cost functions.

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PhD under supervision of Thomas Colcombet

Hebrew University of Jerusalem

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Introduction

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  Toolbox to decide many problems arising naturally.
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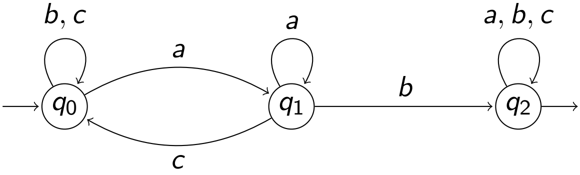
  Theoretical and practical advantages.
- **Problem:**
  Decidability is still open for some automata-related problems.

Descriptions of a language



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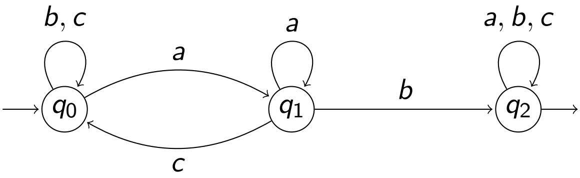

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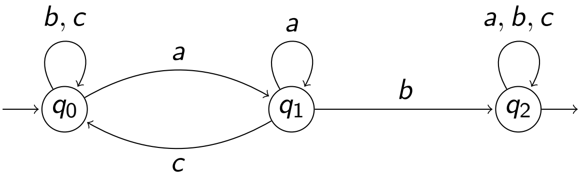

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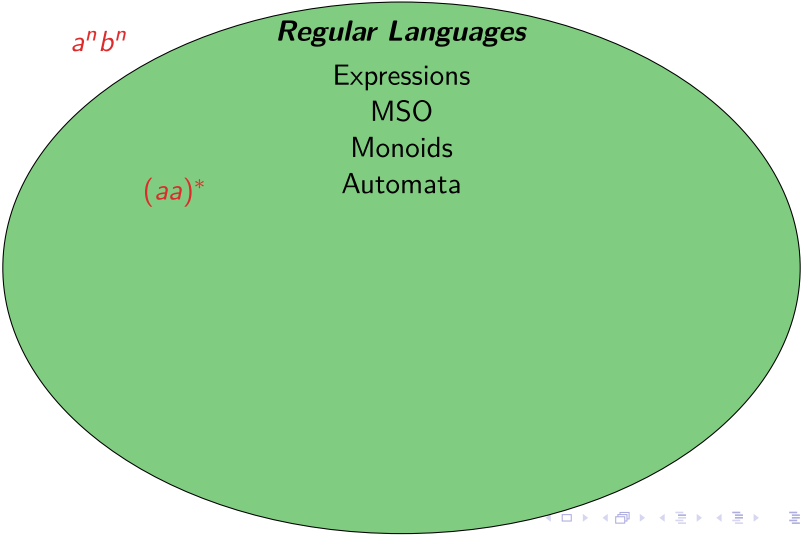
- Logical sentence (MSO) : $\exists x\ \exists y\ a(x) \wedge b(y) \wedge (y = Sx)$.
- Finite monoid : $M = \{1, a, b, c, ba, 0\},\ P = \{0\}$
  $ab = 0,\ aa = ca = a,\ bb = bc = b,\ cc = ac = cb = c$

Regular Languages

All these formalisms are effectively equivalent.



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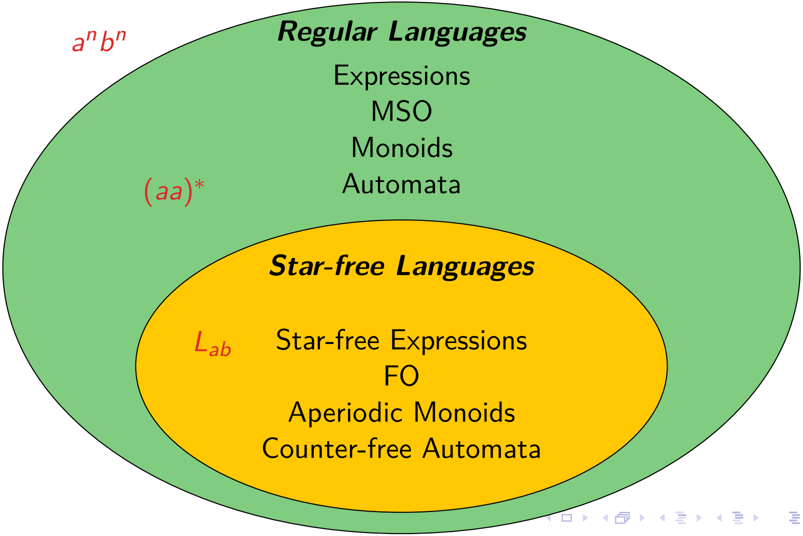

Historical motivation

Given a class of languages $\mathcal{C}$, is there an algorithm which given an automaton for L, decides whether $L \in \mathcal{C}$?

Theorem (Schützenberger 1965)

It is decidable whether a regular language is star-free, thanks to the equivalence with aperiodic monoids.

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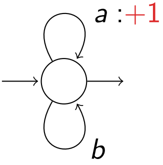
Finite Power Problem: Given L, is there n such that

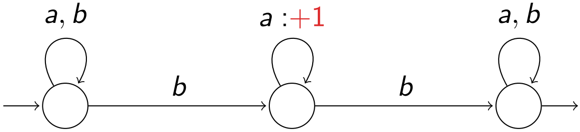
$$(L+\varepsilon)^n = L^* \text{ ?}$$

There is no known algebraic characterization,
other technics are needed to show decidability.

Distance Automata



$\mathcal{A}_1$: number of a



$\mathcal{A}_2$: smallest block of a

Unbounded: There are words with arbitrarily large value.

Deciding **Boundedness** for distance automata $\Rightarrow$ solving finite power problem.

Theorem (Hashiguchi 82, Kirsten 05)

Boundedness is decidable for distance automata.

Problems solved using counters

- ***Finite Power*** (finite words) [Simon '78, Hashiguchi '79]
  Is there n such that $(L+\varepsilon)^n = L^*$?
- ***Fixed Point Iteration*** (finite words)
  [Blumensath+Otto+Weyer '09]
  Can we bound the number of fixpoint iterations in a MSO formula ?
- ***Star-Height*** (finite words/trees)
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]
  Given n, is there an expression for L, with at most n nesting of Kleene stars?
- ***Parity Rank*** (infinite trees)
  [reduction in Colcombet+Löding '08, decidability open, deterministic input Niwinski+Walukiewicz '05]
  Given $i < j$, is there a parity automaton for L using ranks $\{i, i+1, \ldots, j\}$?

Theory of Regular Cost Functions

Aim: General framework for previous constructions.

- Generalize from languages $L : \mathbb{A}^* \to \{0, 1\}$
  to functions $f : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$
- Accordingly generalize automata, logics, semigroups, in order to obtain a theory of regular cost functions, which behaves as well as possible.
- Obtain decidability results thanks to this new theory.

Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$

$+$ finite set of counters
(initialized to 0, values range over $\mathbb{N}$)

$+$ counter operations on transitions
(increment $\mathtt{I}$, reset $\mathtt{R}$, check $\mathtt{C}$, no change ε)

Semantics: $[\![\mathcal{A}]\!] : \Sigma^* \to \mathbb{N} \cup \{\infty\}$

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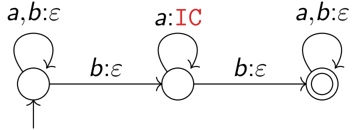
$$val_B(\rho) := \text{max checked counter value during run } \rho$$

$$[\![\mathcal{A}]\!]_B(u) := \min\{val_B(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

Example

$[\![\mathcal{A}]\!]_B(u) =$ min length of block of a's surrounded by b's in u



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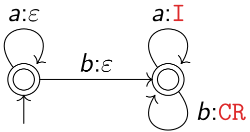

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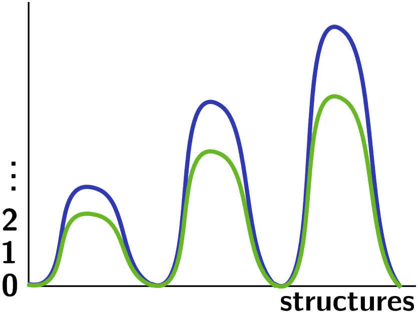
[Colcombet '09, following Bojánczyk+Colcombet '06]
for all subsets U, $[\![\mathcal{A}]\!](U)$ bounded iff $[\![\mathcal{B}]\!](U)$ bounded



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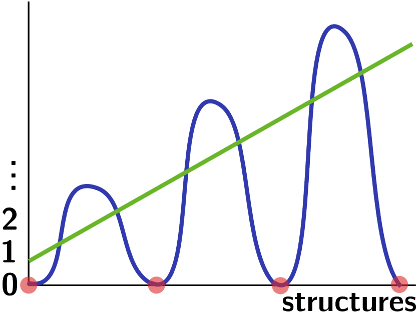

$[\![\mathcal{A}]\!] \not\approx [\![\mathcal{B}]\!]$

Therefore we always identify two functions if they are bounded on the same sets.

Example

For any function f, we have $f \approx 2f \approx \exp(f)$.
But $(u \mapsto |u|_a) \not\approx (u \mapsto |u|_b)$, as witnessed by the set a^*.

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Theorem (Colcombet '09, following Hashiguchi, Leung, Simon, Kirsten, Bojańczyk+Colcombet)

Cost automata $\Leftrightarrow$ Cost logics $\Leftrightarrow$ Stabilisation monoids.
For some suitable models of Cost Logics and Stabilisation Monoids, extending the classical ones.
Boundedness decidable.

All these equivalences are only valid up to $\approx$.
It provides a toolbox to decide boundedness problems.

Languages as cost functions

A language L is represented by its characteristic function

$$\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$$

If $\mathcal{A}$ is a classical automaton for L, then $[\![\mathcal{A}]\!]_B = \chi_L$ and $[\![\mathcal{A}]\!]_S = \chi_{\overline{L}}$. Switching between B and S is the generalization of language complementation.

Cost function theory strictly extends language theory.

All theorems on cost functions are in particular true for languages.

Goal of the thesis: Studying cost function theory, and generalise known theorems from languages to cost functions.

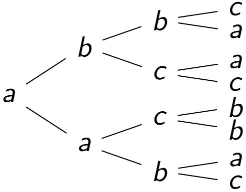
Contributions of the thesis

Input structures:

Finite words: *accba*

Infinite words: *abaabaccbaba* . . .

Infinite trees:



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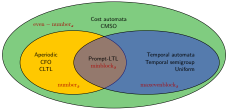

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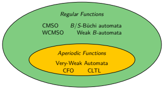

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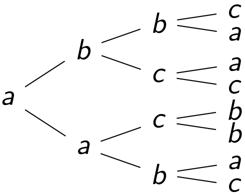

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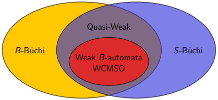
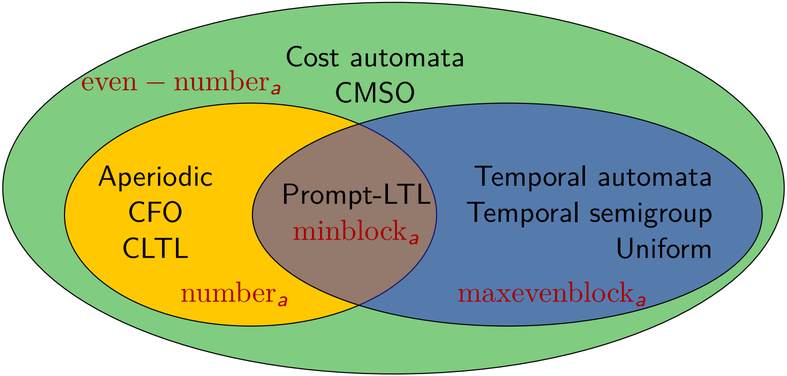


Different kinds of results:

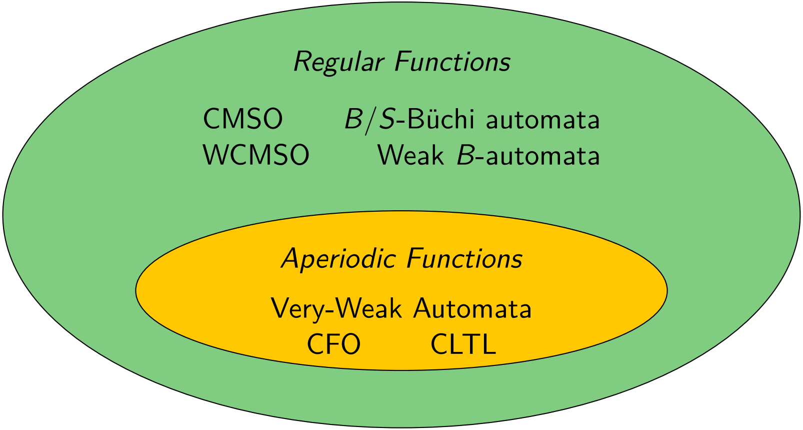
- Generalisation of language notions and theorems,
- Study of classes specific to cost functions,
- Reduction of classical decision problems to boundedness problems.

Cost Functions on finite words



Decidability of membership and effectiveness of translations [Colcombet+K.+Lombardy ICALP '10, K. STACS '11].
Generalization of Myhill-Nerode Equivalence [K. STACS '11].
Boundedness of CLTL is PSPACE-complete [Submitted to LMCS].

Cost Functions on infinite words

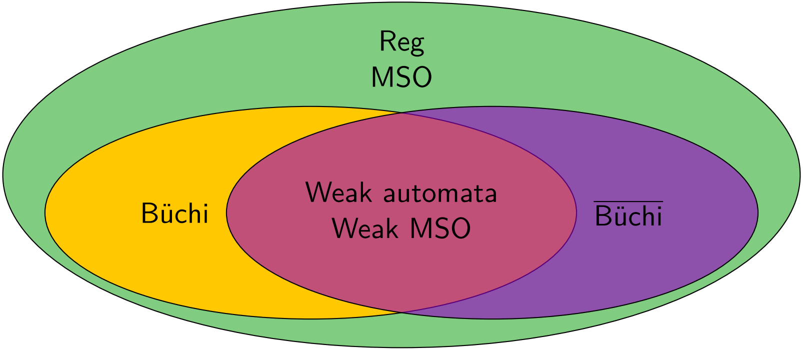


Decidability of membership and effectiveness of translations [K.+Vanden Boom, ICALP '12].

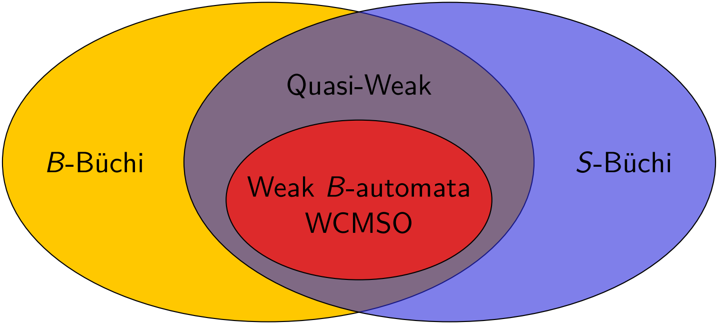
Languages on infinite trees

Theorem (Rabin 1970, Kupferman + Vardi 1999)

L recognizable by an alternating weak automaton $\Leftrightarrow$
L recognizable by WMSO $\Leftrightarrow$ there are Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that $L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}$.



Cost functions on infinite trees

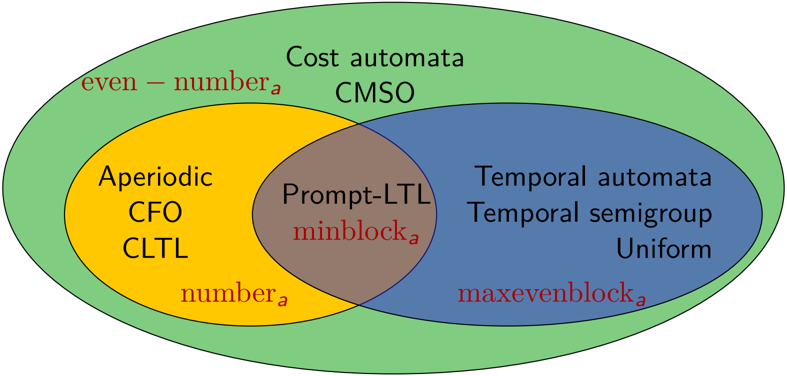


Decidability of boundedness for Quasi-Weak automata.[K.+Vanden Boom, FSTTCS '11].

If $\mathcal{A}$ is a Büchi automaton, it is decidable whether $L(\mathcal{A})$ is weak [submitted to CSL '13].

Logic for the Quasi-Weak class.

Cost Functions on finite words



Logics on Finite Words

- First-Order Logic (FO): we quantify over positions in the word.

$$\varphi := a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \psi \mid \exists x \varphi$$

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- MSO: FO with quantification on sets, noted X, Y.
- Linear Temporal Logic (LTL) over $\mathbb{A}^*$:

$$\varphi := a \mid \Omega \mid \neg\varphi \mid \varphi \vee \psi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \psi$$

$$\varphi\mathbf{U}\psi: \quad \begin{array}{ccccccccccc} \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \psi & & \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array}$$

Future operators **G** (Always) and **F** (Eventually).
Example: To describe L_{ab}, we can write $\mathbf{F}(a \wedge \mathbf{X}b)$.

Generalisation: cost LTL

- CLTL over $\mathbb{A}^*$:

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Negations pushed to the leaves.

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- "Error variable" N is unique, shared by all occurences of $\mathbf{U}^{\leq N}$.
- $\mathbf{G}^{\leq N}\varphi$: φ is false at most N times in the future ($\varphi \mathbf{U}^{\leq N} \Omega$).

Generalisation : Cost FO and Cost MSO

- CFO over $\mathbb{A}^*$:

  $$\varphi := a(x) \mid x = y \mid x < y \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi \mid \forall x \varphi \mid \forall^{\leq N} x \varphi$$

  Negations pushed to the leaves.
- As before, N unique free variable.
- $\forall^{\leq N} x \varphi(x)$ means φ is false on at most N positions.
- CMSO extends CFO by allowing quantification over sets.

Semantics of Cost Logics

From formula to cost function:

Formula $\varphi \longrightarrow$ cost function $[\![\varphi]\!] : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, defined by

$$[\![\varphi]\!](u) = \inf\{n \in \mathbb{N} : \varphi \text{ is true over } u \text{ with } n \text{ as error value}\}$$

Example with the alphabet $\{a, b\}$

- $\mathrm{number}_a = [\![\mathbf{G}^{\leq N} b]\!] = [\![\forall^{\leq N} x \ b(x)]\!]$.

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  $= [\![\forall X \ \mathrm{block}_a(X) \Rightarrow (\forall^{\leq N} x \ x \notin X)]\!]$.

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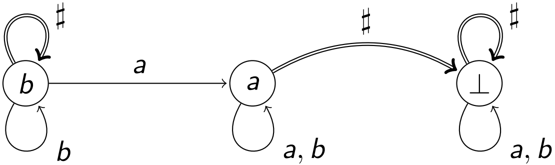
Example: Stabilisation Monoid for number_a

$M = \{b, a, \perp\}$, $P = \{a, b\}$,

b: "no a", a: "a little number of a", $\perp$: "a lot of a".



Cayley graph

Aperiodic Monoids

Definition: A [stabilisation] monoid M is aperiodic if for all $x \in M$ there is $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

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Theorem (McNaughton-Papert, Schützenberger, Kamp)

Aperiodic Monoids $\Leftrightarrow$ *FO* $\Leftrightarrow$ *LTL* $\Leftrightarrow$ *Star-free Expressions.*

We want to generalise this theorem to cost functions.
The problems are:

- No complementation $\Rightarrow$ No Star-free expressions.
- Deterministic automata are strictly weaker.
- Heavy formalisms (semantics of stabilisation monoids).
- New quantitative behaviours.
- Original proofs already hard.

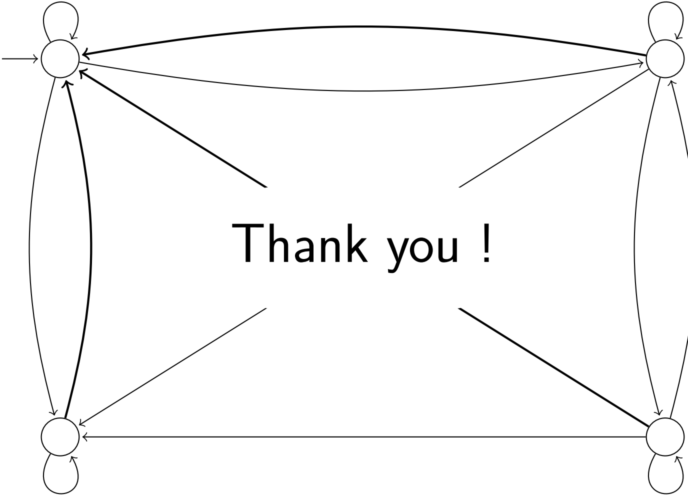
Aperiodic cost functions

Theorem (K. STACS 2011)

Aperiodic stabilisation monoid $\Leftrightarrow$ *CLTL* $\Leftrightarrow$ *CFO.*

Proof Ideas:

- Generalisation of Myhill-Nerode $\Rightarrow$ Syntactic object.
- Induction on $(|M|, |\mathbb{A}|)$.
- Extend functions to sequences of words.
- Use bounded approximations.
- Extend CLTL with Past operators, show Separability.



Thank you !