Computational content of circular proof systems.

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Curry-Howard correspondence

Proof of formula $\varphi \leftrightarrow$ Program of type φ

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Correspondence well-understood for usual proof systems

[Curry,Howard]	Intuitionistic logic	$\leftrightarrow$	Typed λ-calculus
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This work: Study the computational content of cyclic proofs.

Cyclic Proofs

Usual proofs:

$$\dfrac{\text{Axiom}_1 \qquad \dfrac{\text{Axiom}_2}{A}}{B} \qquad \dfrac{\text{Axiom}_3}{C}$$

$$\dfrac{B \qquad C}{D}$$

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As programs: recursive calls must be done on smaller arguments.
$\rightarrow$ guarantees termination.

A proof system for regular expressions

Example: [Das, Pous '17]
Cyclic proof system for inclusion of regular expressions:

$$\dfrac{\dfrac{\dfrac{}{\varepsilon \subseteq \varepsilon}\,(\mathsf{Ax})}{\varepsilon \subseteq a^*}\,(*\text{-right}_1) \qquad \dfrac{\dfrac{}{a \subseteq a}\,(\mathsf{Ax}) \qquad a^* \subseteq a^*}{a, a^* \subseteq a^*}\,(*\text{-right}_2)}{a^* \subseteq a^*}\,(*\text{-left})$$

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Soundness and completeness [Das, Pous '17]

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Here, we care about computational content.

This example: program whose type is $a^* \to a^*$:

```
let rec f l = match l with
                |[] -> []
                |a::q -> a::(f q)
```

Computing languages

Goal: Avoid transductions, start with languages.

- Regular expressions $e, f := a \in A \mid e.f \mid e + f \mid e^*$
- Boolean type bool (encoded by $\varepsilon + \varepsilon$)

Proof of $A^* \vdash \mathit{bool}$ $\Leftrightarrow$ Program of type $A^* \to \mathit{bool}$
$\Leftrightarrow$ Language $L \subseteq A^*$.

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Structural rules: basic data manipulation (erase, copy).

On the proof side: reuse or ignore hypotheses (cf *linear logic*)

Simplified proof system

Expressions $e := A \mid A^*$

Lists $E, F = e_1, e_2, \ldots, e_n$ interpreted as tuples

Proof system:

Return true or false

$$\frac{}{\vdash bool}\ (true) \qquad \frac{}{\vdash bool}\ (false)$$

Pattern matchings

$$\frac{(E, F \vdash bool)_{\underline{a \in A}}}{E, \underline{A}, F \vdash bool}\ \text{(A)} \qquad \frac{E, F \vdash bool \qquad E, A, A^*, F \vdash bool}{E, \underline{A^*}, F \vdash bool}\ (*)$$

Erase, copy

$$\frac{E, F \vdash bool}{E, \underline{e}, F \vdash bool}\ \text{(weakening)} \qquad \frac{E, \underline{e, e}, F \vdash bool}{E, \underline{e}, F \vdash bool}\ \text{(contraction)}$$

Proofs as language acceptors

What are the languages computed by cyclic proofs ?

Example on alphabet $\{a, b\}$: The language b^*

$$\dfrac{\dfrac{}{\vdash bool}\,(\mathit{true}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash bool}\,(\mathit{false})}{(\underline{A^*} \vdash bool)_a}\,(\mathsf{wkn}) \qquad (A^* \vdash bool)_b}{\underline{A}, A^* \vdash bool}\,(\mathsf{A})}{\underline{A^*} \vdash bool}\,(*)$$

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No contraction rule: *Affine* system.

Lemma

The affine system captures exactly regular languages.

With contractions: what class of language?

Example on alphabet $\{a, b\}$: Language $\{a^n b^n \mid n \in \mathbb{N}\}$.

Intuition:

- Copy the input u_1 into u_2: $aaabbb \quad aaabbb$
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Theorem

The proof system recognizes exactly languages in LOGSPACE.

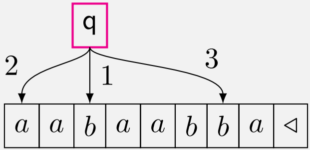
Proof technique: Design an equivalent automaton model.

A new automaton model

Jumping Multihead Automata

A JMA is an automaton with k reading heads.

Transitions: $Q \times (A \cup \{\triangleleft\})^k \to Q \times \{⏭, ◌, J_1, \ldots, J_k\}^k$



- ⏭ : advance one step
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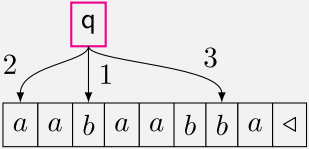

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(optional: Syntactic criterion guaranteeing halting)

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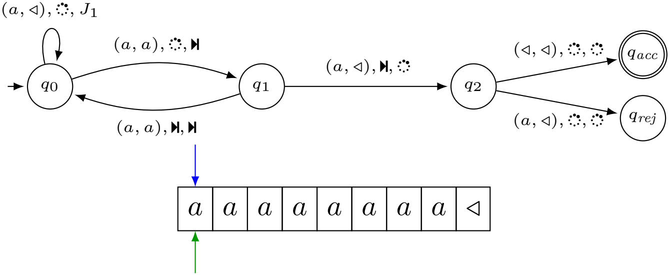
Example: $\{a^{2^n} \mid n \in \mathbb{N}\}$ is accepted by a 2-head JMA.



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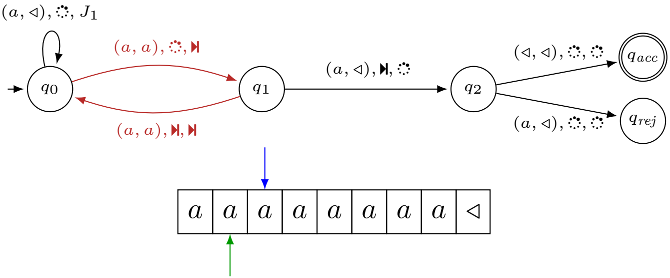

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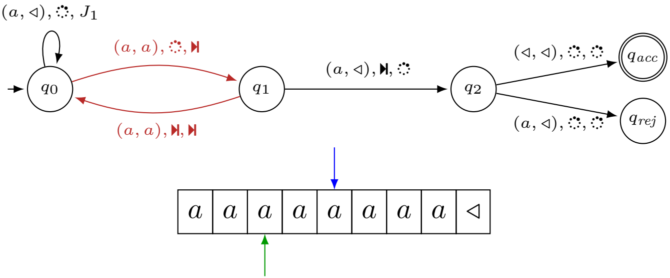

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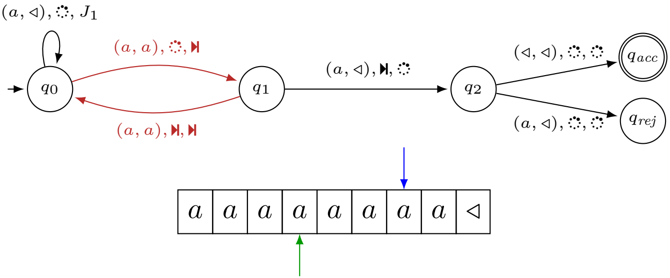

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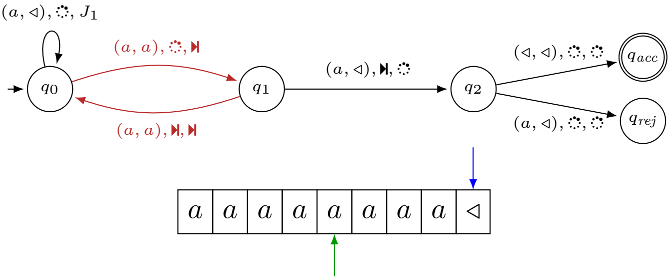

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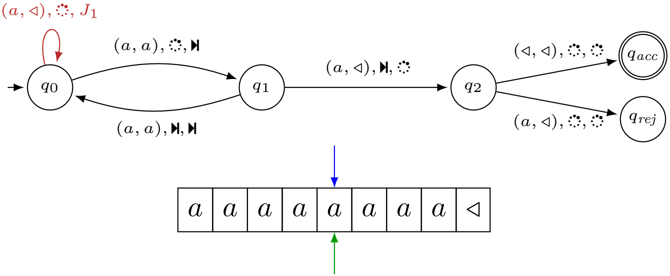

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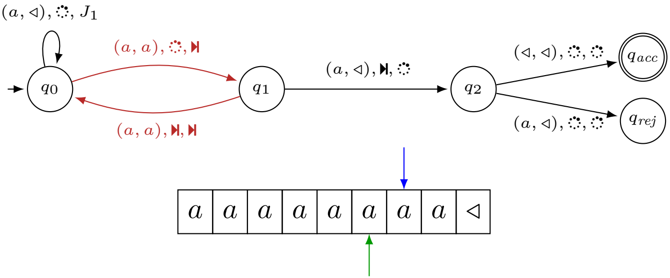

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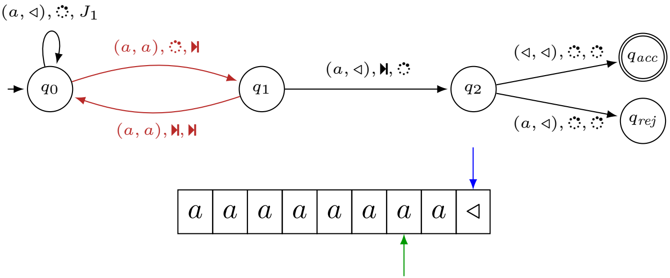

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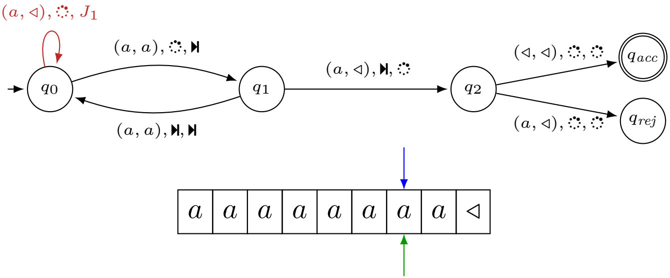

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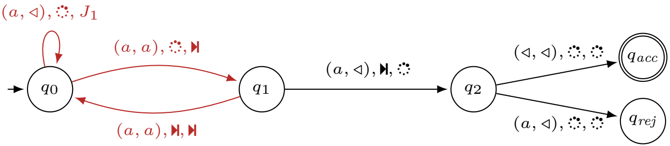

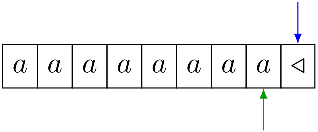

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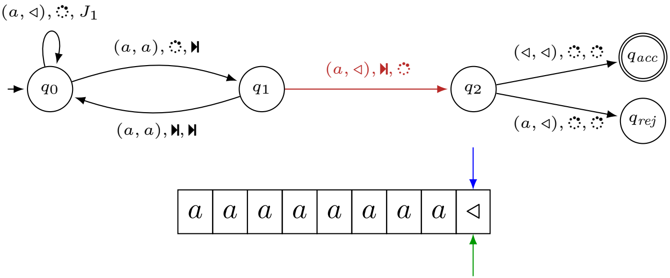

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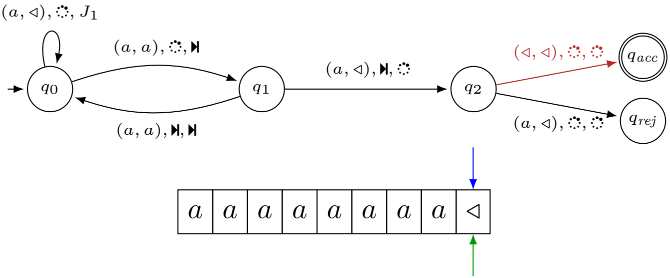

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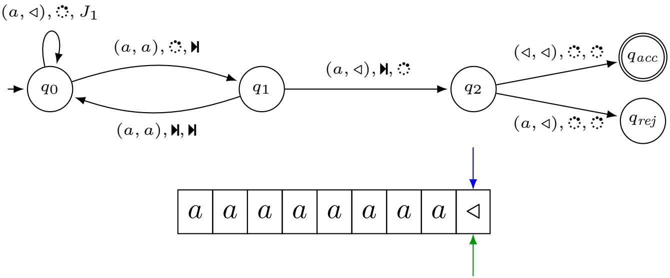

Theorem

Cyclic proofs and JMA recognize the same class of languages.

Expressive power of JMAs

JMAs $\subseteq$ LOGSPACE easy: remember the location of the k heads.

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2MAs: 2-way Multihead Automata [Holzer, Kutrib, Malcher '08]:

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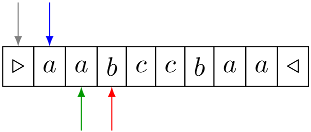

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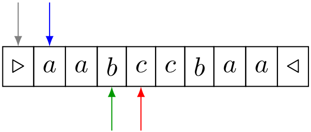

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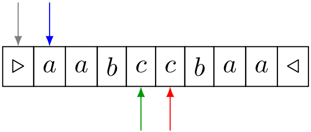

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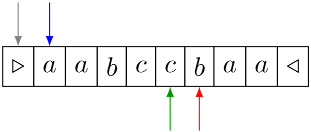

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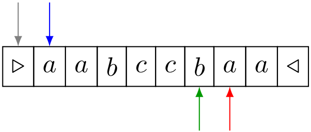

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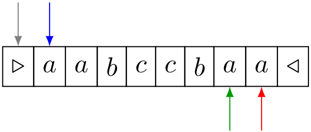

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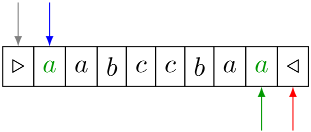

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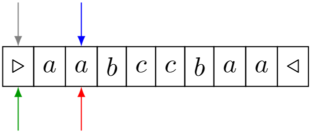

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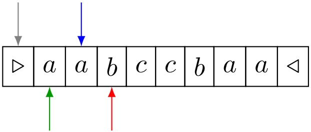

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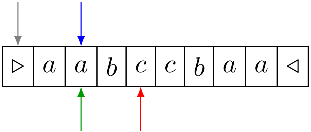

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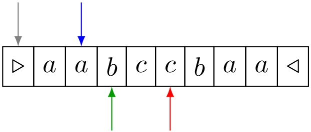

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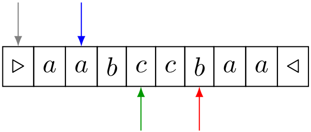

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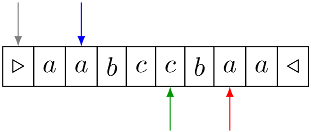

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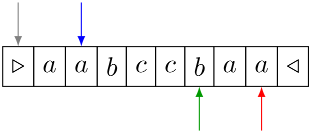

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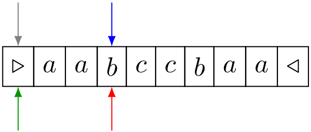

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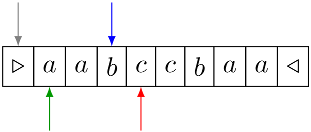

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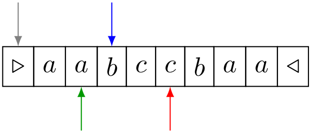

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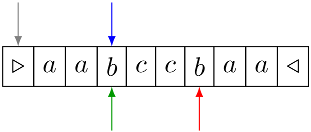

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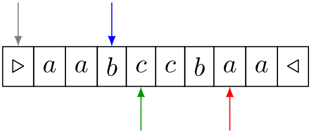

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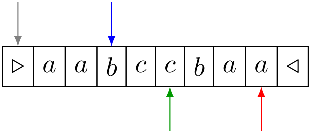

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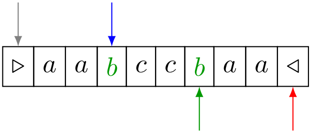

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Generalization of this idea $\Rightarrow$ Translation from 2MA to JMA.

Cut rule and integer functions

The cut rule:

$$\frac{E \vdash e \qquad e, F \vdash g}{E, F \vdash g}$$

- Corresponds to composition of programs.
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Which functions $\mathbb{N}^k \to \mathbb{N}$ can the system with cuts compute ?

System T

As automata before, we want a computational framework to characterize the expressive power of our cyclic proof system.

System T:

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(+constructors/destructors for pairs, lists)

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Example: Addition $a + b$: $\lambda ab.\mathbf{Rec}(b, a, s(y))$

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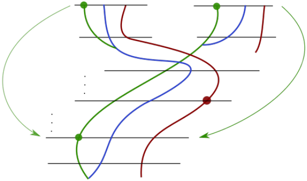

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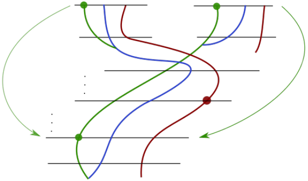

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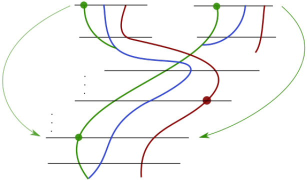

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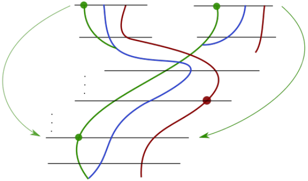

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Open problems in proof theory: infinite descent versus induction.

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Proofs $\to$ System T:

- ACA_0: RCA_0 + König's lemma,
- $\forall$ cyclic proof , prove in ACA_0 that its computation terminates,
- Conservativity result: $\mathsf{ACA}_0 \leftrightarrow$ Peano for integer functions,
- Classic result: Peano $\leftrightarrow$ System T.

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