Bouncing threads for infinitary and circular proofs

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The logic $\mu$MLL

Syntax of $\mu$MLL

$$\varphi, \psi := a | a^\perp | \bot | 1 | X | \varphi \otimes \psi | \varphi \otimes \psi | \mu X.\varphi | \nu X.\varphi$$

No negation in the syntax, but dualization operation $\varphi \mapsto \varphi^\perp$ with $(\varphi^\perp)^\perp = \varphi$.

- $X^\perp = X$
- $\otimes$ disjunction, $\otimes$ conjunction
- $\mu$ least fixpoint, $\nu$ greatest fixpoint
- $\mu X. X$ false, $\nu X. X$ true
### Syntax of $\mu$MLL

$$\varphi, \psi := a \mid a^\perp \mid \bot \mid 1 \mid X \mid \varphi \band \psi \mid \varphi \boxtimes \psi \mid \mu X.\varphi \mid \nu X.\varphi$$

No negation in the syntax, but dualization operation $\varphi \mapsto \varphi^\perp$ with $(\varphi^\perp)^\perp = \varphi$.

- $X^\perp = X$
- $\band$ disjunction, $\boxtimes$ conjunction
- $\mu$ least fixpoint, $\nu$ greatest fixpoint
- $\mu X.\varphi$ false, $\nu X.\varphi$ true

Motivation: proofs as programs.
Proof system

Rules of \( \mu \text{MLL} \)

\[
\begin{align*}
\Gamma & \quad \vdash \Gamma & \quad (\bot) \\
\vdash F, G, \Gamma & \quad \vdash F \otimes G, \Gamma & \quad (\otimes) \\
\vdash F[\mu X.F/X], \Gamma & \quad \vdash \mu X.F, \Gamma & \quad (\mu) \\
\vdash G[\nu X.G/X], \Gamma & \quad \vdash \nu X.G, \Gamma & \quad (\nu) \\
\vdash F, F^\bot & \quad (\text{Ax}) \\
\vdash \Gamma, F & \quad \vdash F^\bot, \Delta & \quad (\text{Cut}) \\
\vdash \Gamma, \Delta & \quad (1)
\end{align*}
\]
Infinite proofs

We allow infinite proofs:

\[
\begin{align*}
\vdash & \mu X.X \\
\vdash & \mu X.X \quad (\mu) \\
\vdash & \mu X.X \\
\vdash & \nu X.X \quad (\nu) \\
\vdash & \nu X.X
\end{align*}
\]
Infinite proofs

We allow infinite proofs:

\[
\begin{align*}
\vdash & \, \mu X.X \\
\vdash & \, (\mu) \\
\vdash & \, \mu X.X \\

\vdash & \, \nu X.X \\
\vdash & \, (\nu) \\
\vdash & \, \nu X.X
\end{align*}
\]

However, only the right one should be valid. We add a validity condition.
Threads

Rules with threads

\[
\begin{align*}
\frac{}{\Gamma \vdash \bot, \Gamma} \quad (\bot) \\
\frac{\Gamma \vdash \bot, \Gamma}{\vdash \bot, \Gamma} \quad (1) \\
\frac{\Gamma \vdash F, G, \Gamma}{\vdash F \otimes G, \Gamma} \quad (\otimes) \\
\frac{\Gamma \vdash F, \Gamma \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta} \quad (\otimes) \\
\frac{\Gamma \vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma} \quad (\mu) \\
\frac{\Gamma \vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma} \quad (\nu) \\
\frac{}{\Gamma, F \vdash F \perp} \quad (Ax) \\
\frac{\Gamma \vdash F, \Gamma \vdash F \perp, \Delta}{\Gamma, \Delta \vdash} \quad (Cut)
\end{align*}
\]
Infinite threads, validity

Example with $F = \nu X. (X \otimes (\mu Y. X))$.

\[
\begin{array}{c}
\vdash F \\
\vdash F \\
\vdash F \\
\vdash F \otimes \mu Y. F \\
\vdash F \otimes \mu Y. F \\
\vdash F \otimes \mu Y. F \\
\vdash F \\
\vdash F \\
\end{array}
\]

(\mu) (\otimes) (\nu)
Infinite threads, validity

Example with \( F = \nu X.(X \otimes (\mu Y . X)) \).

\[
\begin{array}{c}
\vdash F \\
\vdash F \\
\vdash \mu Y . F \\
\vdash \mu Y . F \\
\vdash F \otimes \mu Y . F \\
\vdash F \otimes \mu Y . F \\
\vdash F \otimes \nu Y . F \\
\vdash F \otimes \nu Y . F \\
\vdash F \\
\end{array}
\]

A thread is **valid** if it sees infinitely many \( \nu \)-unfoldings (roughly).
Infinite threads, validity

Example with $F = \nu X. (X \otimes (\mu Y.X))$.

$\vdash F$ \\
$\vdash F$ \\
$\vdash F \otimes (\mu Y.F)$ \\
$\vdash F \otimes \mu Y.F$ \\
$\vdash F$ \\

A thread is valid if it sees infinitely many $\nu$-unfoldings (roughly).

A proof is valid if every infinite branch contains a valid thread.
Infinite threads, validity

Example with $F = \nu X.(X \otimes (\mu Y.X))$.

A thread is valid if it sees infinitely many $\nu$-unfoldings (roughly).

A proof is valid if every infinite branch contains a valid thread.

Theorem (BDS 2016)

This system is sound, and admits cut-elimination
A proof with cut

Problem: Cuts are not well-managed by the validity condition.

\[ \frac{\vdash \nu X.X, \mu X.X}{\vdash \nu X.X, \mu X.X} \quad \text{(Ax)} \]

\[ \frac{\vdash \nu X.X, \mu X.X}{\vdash \nu X.X} \quad \text{(Cut)} \]

\[ \frac{\vdash \nu X.X}{\vdash \nu X.X} \quad \text{(Cut-elimination)} \]
A proof with cut

Problem: Cuts are not well-managed by the validity condition.

\[
\frac{\vdash \nu X.X, \mu X.X}{\vdash \nu X.X, \mu X.X} \quad (Ax) \\
\frac{\vdash \nu X.X, \mu X.X}{\vdash \nu X.X} \quad (\mu) \\
\frac{\vdash \nu X.X, \mu X.X}{\vdash \nu X.X} \quad (\nu) \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \quad (Cut) \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \quad (Cut) \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \quad (\nu) \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \quad (\nu) \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \\
\frac{\vdash \nu X.X}{\vdash \nu X.X} \quad (Cut-elimination)
\]
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

\[
\begin{align*}
\vdash F, F \perp & \quad \text{(Ax)} \quad \vdash G, G \perp & \quad \text{(Ax)} \quad \vdash F', G & \quad \nu \\
\vdash F \otimes G, F \perp \otimes G & \quad \text{(\otimes)} \quad \vdash F, G \perp & \quad \otimes \quad \vdash F \otimes G & \quad \text{Cut} \\
\end{align*}
\]
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

\[
\begin{align*}
\vdash F \perp, F \perp & \quad (Ax) \\
\vdash G \perp, G \perp & \quad (Ax) \\
\vdash F \otimes G, F \perp \otimes G \perp & \quad (\otimes, \otimes) \\
\vdash F \otimes G & \quad (\otimes) \\
\end{align*}
\]
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

$\vdash F, F \bot$

$\vdash G, G \bot$

$\vdash F \otimes G, F \bot \otimes G \bot$

$\vdash F \otimes G$

$\vdash F', G$

$\vdash F, G$

$\vdash F \otimes G$

$\vdash F' \otimes G$

$(\nu)$

$(\otimes)$

$(\text{Cut})$

$(x, \bot)$

$\bot$

$(x, x) \mid \text{pop}()$

$(\bar{x}, \gamma) \mid \text{push}(x)$

Axiom

Cut

left
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F, G$ arbitrary:
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

\[
\frac{\vdash F, F \bot}{\vdash F \otimes G, F \bot \otimes G} \quad (\text{Ax})
\]

\[
\frac{\vdash G, G \bot}{\vdash F \otimes G} \quad (\otimes, \otimes)
\]

\[
\vdash F, G \quad (\nu)
\]

\[
\vdash F', G \quad (\otimes)
\]

\[
\vdash F \otimes G \quad (\text{Cut})
\]
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

\[
\begin{align*}
\vdash F, F \perp & \quad \text{(Ax)} \\
\vdash G, G \perp & \quad \text{(Ax)} \\
\vdash F \otimes G, F \perp \otimes G & \quad \text{(\otimes, \otimes)} \\
\vdash F \otimes G & \quad \text{(Cut)}
\end{align*}
\]
Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with $F$, $G$ arbitrary:

\[
\begin{array}{c}
\vdash F, F \perp \\
\vdash G, G \perp \\
\vdash F \otimes G, F \perp \otimes G \perp \\
\vdash F \otimes G
\end{array}
\]

\[
\begin{array}{c}
(\text{Ax}) \\
(\text{Ax}) \\
(\otimes) \\
(\otimes) \\
(\text{Cut})
\end{array}
\]

\[
\begin{array}{c}
(x, x) \mid \text{pop()} \\
(\bar{x}, \gamma) \mid \text{push}(x)
\end{array}
\]

\[
\begin{array}{c}
\text{Axiom} \\
\text{Cut}
\end{array}
\]

left, $\nu$
Bouncing threads: visible part

Visible part: survives the cut-elimination.
Hidden part: Must satisfy matching constraints.

Bouncing thread valid: $\infty \nu$-unfoldings in visible part.
Valid branches

Branch \( B \text{ valid} \): \( \exists \) valid thread with visible part included in \( B \).

Proof valid: all infinities branches are valid.
Reason: guarantee productivity of cut-elimination algorithm.

Theorem

Soundness and cut-elimination still hold.
Decidability of the validity condition?

Given a circular proof, decide validity?
Decidability of the validity condition?

Given a circular proof, decide validity?

Answer: NO.
Decidability of the validity condition?

Given a **circular** proof, decide validity?

Answer: **NO**.

Ingredients:

- **Minsky Machines**: 2-counter finite-state deterministic machine with Increment, Decrement, Zero test
- Simulate the run of a machine with a bouncing thread on $F = \nu X.(X \otimes X)$
- If terminates, rewind computation to erase extra constraints
Zooming on the encoding

Encoding details:

- \( F = \nu X . (X \otimes X) \), \( G = F^\perp = \mu X . (X \otimes X) \).
- Current state \( p \) encoded in node of the graph.
- Counter values \( n, m \) encoded in a constraint stack \( I^n r l^m r \).

Example: **Encoding** \( p \xrightarrow{\text{Inc}_1} q \)

\[
\begin{align*}
\vdash G_l, F & \quad \text{(Ax)} & \vdash G_r & \quad \text{(\( \infty \))} \\
\vdash G_l \otimes G_r, F & \quad \text{(\( \otimes \))} & \vdash G, F & \quad \text{(\( \mu \))} \\
(q) \vdash G, F & \quad \text{and} & \vdash G, F & \quad \text{(Cut)} \\
\vdash G_l, F & \quad \text{(Ax)} & \vdash G_r & \quad \text{(\( \infty \))} \\
\vdash G_l \otimes G_r, F & \quad \text{(\( \otimes \))} & \vdash G, F & \quad \text{(\( \mu \))} \\
(p) \vdash G, F & \quad \text{(Cut)}
\end{align*}
\]

\[ I^n r l^m r \leftrightarrow I^{n+1} r l^m r \]
A hierarchy of decidable conditions

Height of a bouncing thread: parameter \( \approx \) “memory”.

Height = maximal height of the stack of the pushdown automaton.

\( k \)-proof: valid proof using only threads of height \( \leq k \).

Theorem

Every valid circular proof is a \( k \)-proof for some \( k \in \mathbb{N} \).

Theorem

For all \( k \in \mathbb{N} \), it is decidable whether a circular proof is a \( k \)-proof.