Kleene algebra with hypotheses

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Kleene Algebra

Regular expressions

$$e, f := a \in \Sigma \mid 1 \mid e + f \mid e \cdot f \mid e^*$$

Intuition : $x^* = \bigcup_{i \in \mathbb{N}} x^i$.
$x \leq y$ shorthand for $x + y = y$

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$(M, +, \cdot, *, 0, 1)$ is a Kleene algebra if:

Axioms of KA

$(M, +, \cdot, 0, 1)$ is an idempotent semiring

$$1 + xx^* \leq x^* \qquad xy \leq y \;\Rightarrow\; x^*y \leq y$$

$$1 + x^*x \leq x^* \qquad yx \leq y \;\Rightarrow\; yx^* \leq y$$

Some concrete Kleene algebras

Models of KA:

- Formal languages: $(\Sigma^*, \cup, \cdot, *, \emptyset, \{\epsilon\})$
- Regular languages: $(\mathrm{Reg}, \cup, \cdot, *, \emptyset, \{\epsilon\})$
- Relations on a set S: $(\mathcal{P}(S \times S), \cup, \circ, *, \emptyset, id)$
- Tropical algebra: $(\mathbb{R} \cup \{-\infty\}, \min, +, *, -\infty, 0)$

  interpret shortest path algorithm
- $n \times n$ matrices over a KA K: $(M_n(K), +, \cdot, *, 0, Id)$

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Theorem (Boffa '90+Krob '91, Kozen '94)

The model $(\mathrm{Reg}_, +, \cdot, *, \emptyset, \{\epsilon\})$ *of regular languages is complete:*

$$KA \vdash e \leq f \Leftrightarrow L(e) \subseteq L(f)$$

Consequence:

Equational theory of KA is decidable and PSPACE-complete.

KA for verification

Programs can be modeled by Relations on S: memory states.
$+$ is nondeterministic choice, $\cdot$ is sequential composition, $*$ is loop

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Example: A : x := 1
B : y := 2

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~~Equational theory of KA~~
Horn Theory of KA $\equiv$ equational theory of KA_H

Star continuity

Intended meaning of star: $x^* = \bigcup_{i \in \mathbb{N}} x^i$.

KA*: KA + this axiom:

Star continuity axiom

$$(\forall i \in \mathbb{N},\ xy^i z \leq t) \quad \Rightarrow \quad xy^* z \leq t$$

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Horn KA* algorithmic problem:
Input: set $H = \{e_i \leq f_i, i \in I\}$ of hypotheses, and target $e \leq f$
Output: Does $KA^*_H \models e \leq$ f hold ?

Research Program [Kozen, Cohen, Conway]
Complexity depending on restrictions on H?

Previous results

Notations for H: a, b: letters; u, v words; e, f expressions.

Decidable cases, if equations in H are of the form:

- $e = 0$ [Cohen 94]
- $a \leq 1$: models **tests** [Cohen 94]

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

- $1 = u$ or $a = u$, (under certain conditions),
  examples: $1 = aa \quad a = aa$ [Kozen+Mamouras 2014]

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Useful for case analysis

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Undecidable cases (Cohen '94, Kozen 2002)

- $ab = ba$: Π^0_1-complete (KA: EXPSPACE-hard)
- $u = v$: Π^0_2-complete (KA: Σ^0_1-complete)
- $e \leq f$: Π^1_1-complete (KA: Σ^0_1-complete)

This paper

Our goal:

- Understand better the decidability frontier
- Stengthen Kozen's hardness results
- Solve Cohen's open problem $1 = \sum_i a_i$
- Study related cases $a \leq \sum_i a_i$ and $a \leq \sum_i u_i$

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From now on, hypotheses are *simple*: $a \leq \sum_i u_i$

H-closure of languages

H-closure of a language L

$cl_H(L)$ is the smallest language such that

- $L \subseteq cl_H(L)$
- $\left(\forall i \in I,\ x u_i y \in cl_H(L)\right) \Rightarrow x a y \in cl_H(L)$

$a \leq \sum_{i \in I} u_i$ hypothese in H, x, y arbitrary words

Example: $L = \{ant, bat, cat, rat\}$, H :

$$e \leq t \quad (1)$$
$$p \leq n + a \quad (2)$$
$$a \leq b + c + r \quad (3)$$

Do we have $ape \in \mathrm{cl}_H(L)$?

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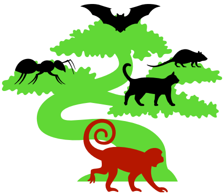

$$\dfrac{\dfrac{ant \qquad \dfrac{bat \quad cat \quad rat}{aat}\,(3)}{apt}\,(2)}{ape}\,(1)$$

Relating closure and Kleene algebra

Theorem

For all hypotheses H, and expressions e, f:

$$\mathsf{KA}_H^* \models e \leq f \qquad \Leftrightarrow \qquad L(e) \subseteq \mathrm{cl}_H(L(f))$$

Is $cl_H(L)$ regular when L regular ?

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$a \leq \sum_i a_i$	No	KA^*_H undecidable for $a \leq \sum_i a_i$.

Our results

Decidability of Cohen's hypotheses $1 = \sum_i a_i$

	$a \leq \sum b$	$a \leq \sum w$	$a \leq f$
$\mathsf{KA}_H \vdash u \leq f$	$\mathrm{EXPTIME-c}$	Σ^0_1−complete	Σ^0_1−complete
$\mathsf{KA}_H \vdash e \leq f$	Undecidable	Σ^0_1−complete	Σ^0_1−complete
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$\mathsf{KA}^*_H \vdash e \leq f$	Π^0_1−complete	Π^0_2−complete	Π^1_1−complete

Ideas: Encode LBA, Turing machines

Recursion-theoretic arguments

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$KA_H \vdash e \leq f$	Undecidable	Σ_1^0−complete	Σ_1^0−complete
$KA_H^* \vdash u \leq f$	$\mathrm{EXPTIME} - \mathrm{c}$	Σ_1^0−complete	Π_1^1−complete
$KA_H^* \vdash e \leq f$	Π_1^0−complete	Π_2^0−complete	Π_1^1−complete

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Future work:

- Refine bounds
- Link with recent works
  e.g. Kuznetsov '19: The logic of action lattices is undecidable
- Coq implementations