Computational content of circular proof systems.

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LIP, ENS Lyon

Lambda Pros
Paris, June 28th 2023

1 Context

2 Computing languages

3 Computing functions

Curry-Howard correspondence

Proof of formula $\varphi \leftrightarrow$ Program of type φ

Example: The identity program $\lambda x.x$ is a proof of $p \rightarrow p$.

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Deduction Rule

$$\text{implies}\Big\downarrow \qquad \frac{A \qquad\qquad\qquad B}{C}$$

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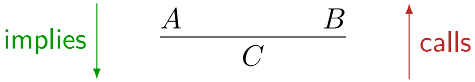
Program instruction



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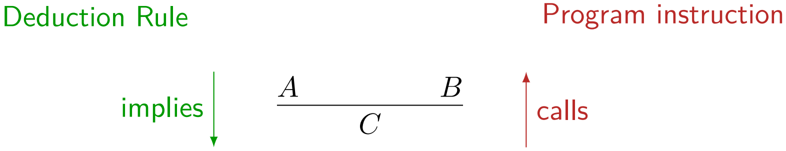
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Correspondence well-understood for usual proof systems

[Curry,Howard]	Intuitionistic logic	$\leftrightarrow$	Typed λ-calculus
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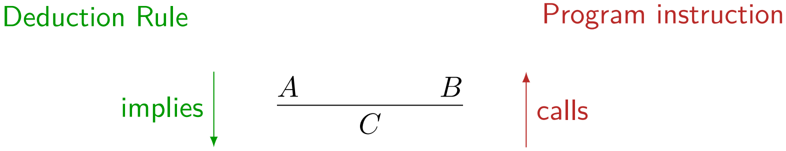

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This work: Study the computational content of cyclic proofs.

Cyclic Proofs

Usual proofs:

$$\dfrac{\dfrac{\text{Axiom}_1 \qquad \dfrac{\text{Axiom}_2}{A}}{B} \qquad \dfrac{\text{Axiom}_3}{C}}{D}$$

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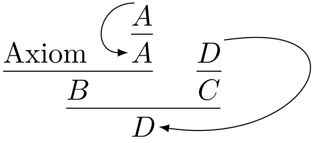

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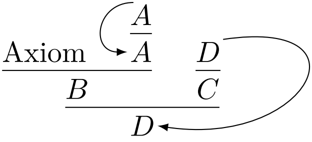

Validity conditions: Cycles must contain particular rules.

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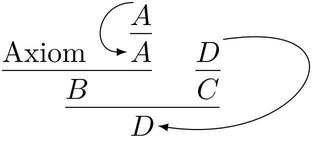

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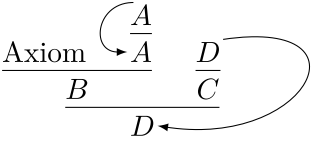
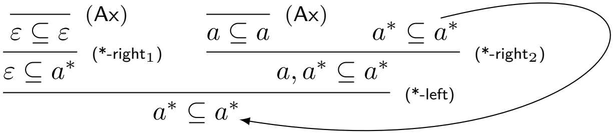


Validity conditions: Cycles must contain particular rules.

As programs: recursive calls must be done on smaller arguments.
→ guarantees termination.

A proof system for regular expressions

Example: [Das, Pous '17]
Cyclic proof system for inclusion of regular expressions:



$$\dfrac{\dfrac{\dfrac{}{\varepsilon \subseteq \varepsilon}\,(\mathsf{Ax})}{\varepsilon \subseteq a^*}\,(\text{*-right}_1) \qquad \dfrac{\dfrac{}{a \subseteq a}\,(\mathsf{Ax}) \qquad a^* \subseteq a^*}{a, a^* \subseteq a^*}\,(\text{*-right}_2)}{a^* \subseteq a^*}\,(\text{*-left})$$

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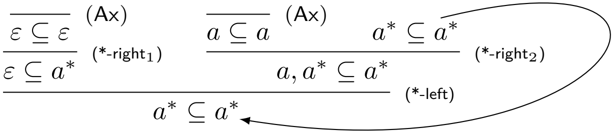

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Soundness and completeness [Das, Pous '17]

$L(e) \subseteq L(f) \Leftrightarrow \exists$ proof of $e \subseteq f$.

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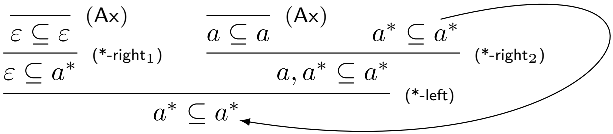

Soundness and completeness [Das, Pous '17]

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Here, we care about computational content.

This example: program whose type is $a^* \to a^*$:

```
let rec f l = match l with
                |[] -> []
                |a::q -> a::(f q)
```

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Computing languages

Goal: Avoid transductions, start with languages.

- Regular expressions $e, f := a \in A \mid e.f \mid e + f \mid e^*$
- Boolean type $bool$ (encoded by $\varepsilon + \varepsilon$)

Proof of $A^* \vdash bool$ $\Leftrightarrow$ Program of type $A^* \to bool$
$\Leftrightarrow$ Language $L \subseteq A^*$.

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Structural rules: basic data manipulation (erase, copy).

On the proof side: reuse or ignore hypotheses (cf *linear logic*)

Simplified proof system

Expressions $e := A \mid A^*$

Lists $E, F = e_1, e_2, \ldots, e_n$ interpreted as tuples

Proof system:

Return true or false

$$\frac{}{\vdash bool}\ (true) \qquad \frac{}{\vdash bool}\ (false)$$

Pattern matchings

$$\frac{(E, F \vdash bool)_{\underline{a \in A}}}{E, \underline{A}, F \vdash bool}\ (\mathsf{A}) \qquad \frac{E, F \vdash bool \qquad E, A, A^*, F \vdash bool}{E, \underline{A^*}, F \vdash bool}\ (*)$$

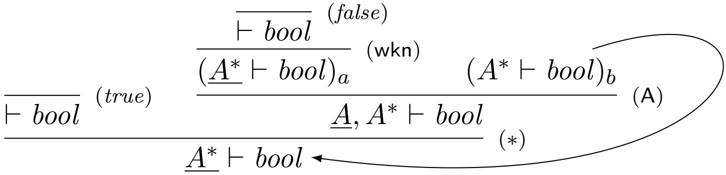
Erase, copy

$$\frac{E, F \vdash bool}{E, \underline{e}, F \vdash bool}\ (\text{weakening}) \qquad \frac{E, \underline{e, e}, F \vdash bool}{E, \underline{e}, F \vdash bool}\ (\text{contraction})$$

Proofs as language acceptors

What are the languages computed by cyclic proofs ?

Example on alphabet $\{a, b\}$: The language b^*



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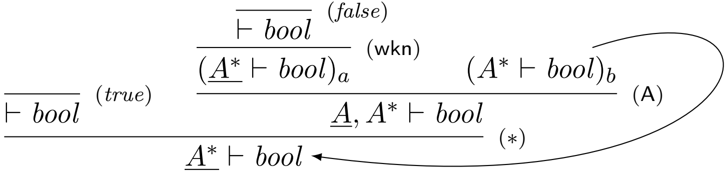

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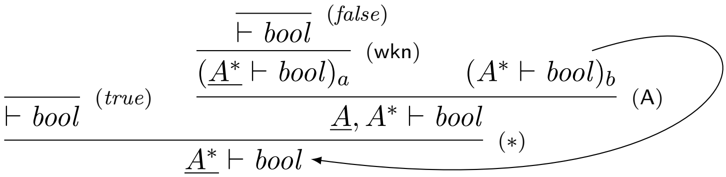

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No contraction rule: *Affine* system.

Lemma

The affine system captures exactly regular languages.

With contractions: what class of language?

Example on alphabet $\{a, b\}$: Language $\{a^n b^n \mid n \in \mathbb{N}\}$.

Intuition:

- Copy the input u_1 into u_2: $aaabbb \quad aaabbbb$
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Theorem

The proof system recognizes exactly languages in LOGSPACE.

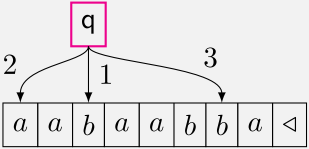
Proof technique: Design an equivalent automaton model.

A new automaton model

Jumping Multihead Automata

A JMA is an automaton with k reading heads.

Transitions: $Q \times (A \cup \{\triangleleft\})^k \to Q \times \{\blacktriangleright\!|, ◌, J_1, \ldots, J_k\}^k$



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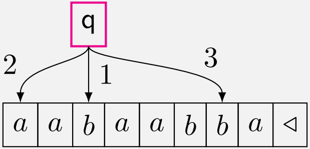

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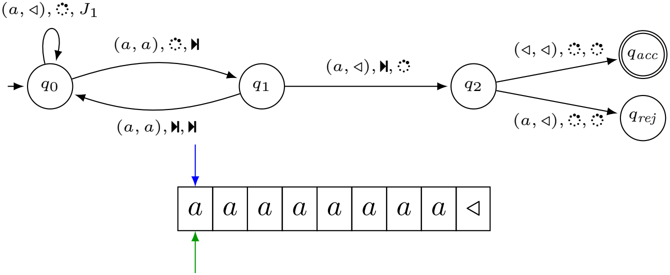
(optional: Syntactic criterion guaranteeing halting)

Example of JMA

Example: $\{a^{2^n} \mid n \in \mathbb{N}\}$ is accepted by a 2-head JMA.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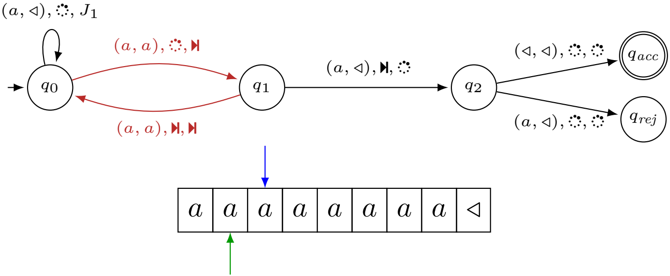

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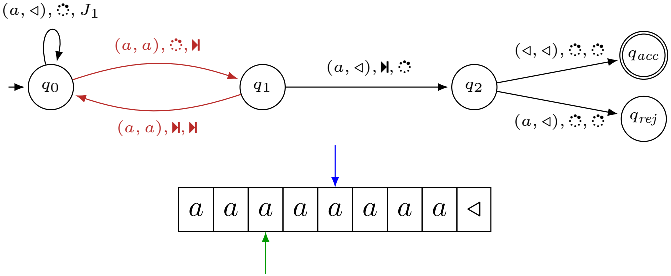
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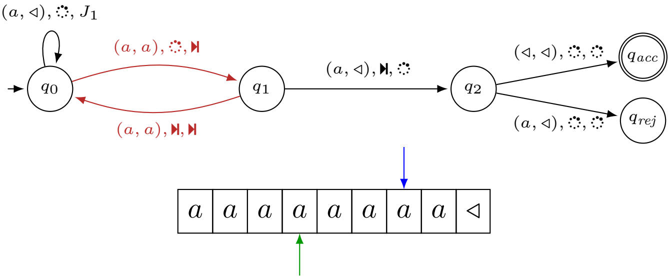
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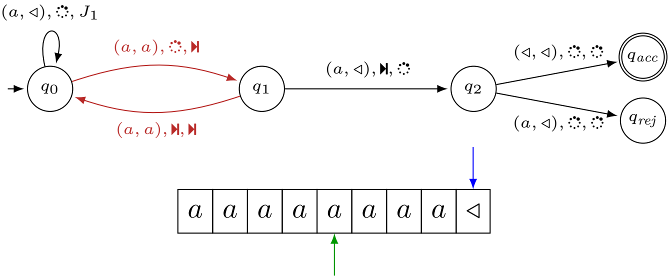
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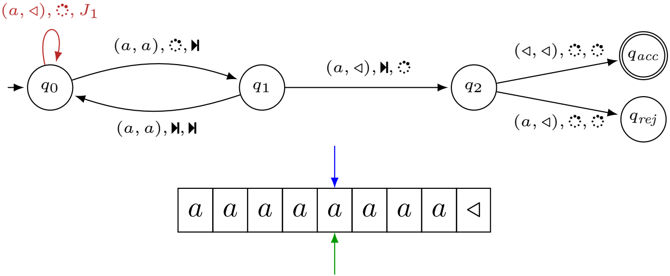
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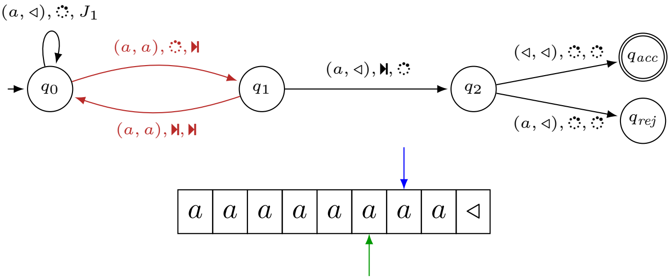
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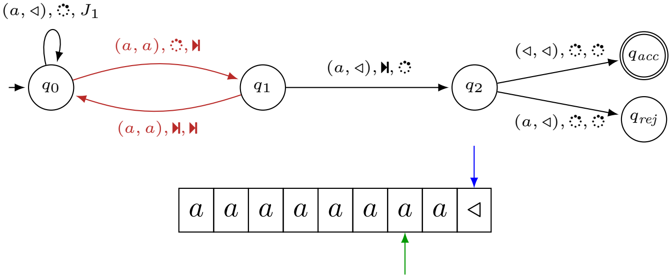

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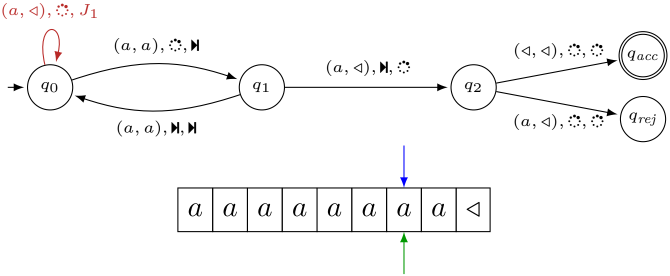
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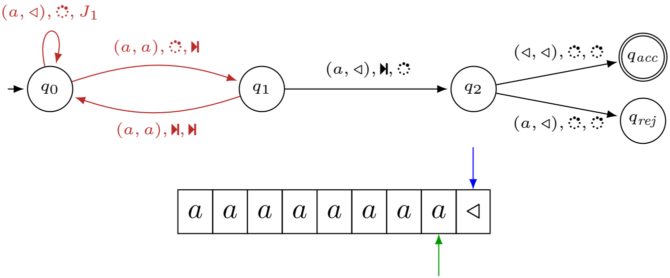

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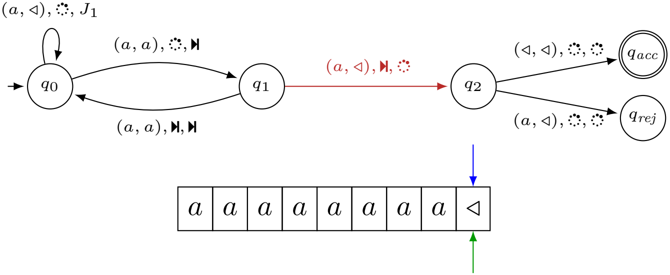
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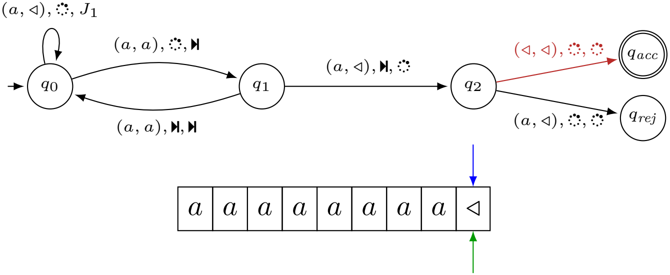

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ACCEPT

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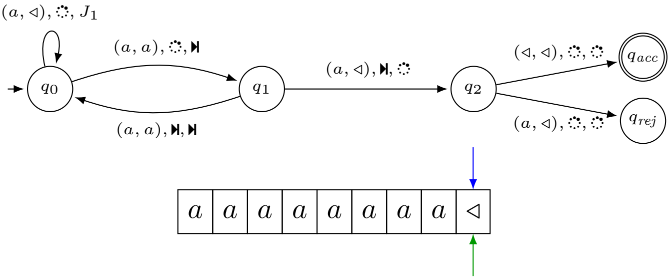
$(a, \triangleleft), ⁘, J_1$
$(a, a), ⁘, ⧐$
$(a, a), ⧐, ⧐$
$(a, \triangleleft), ⧐, ⁘$
$(\triangleleft, \triangleleft), ⁘, ⁘$
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q_0
q_1
q_2
q_{acc}
q_{rej}
a a a a a a a a $\triangleleft$

Theorem

Cyclic proofs and JMA recognize the same class of languages.

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JMAs $\subseteq$ Logspace easy: remember the location of the k heads.

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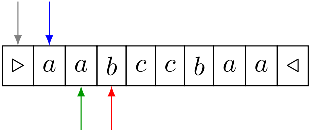

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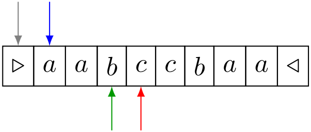

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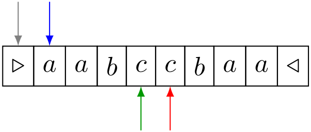

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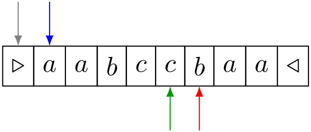

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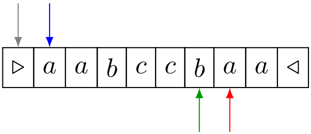

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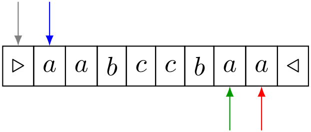
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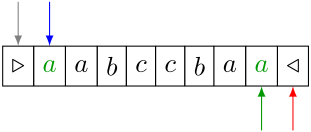
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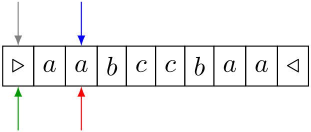

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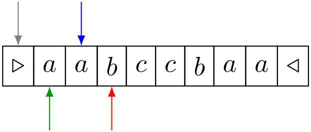

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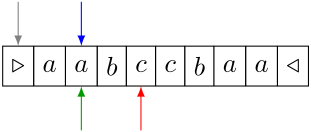

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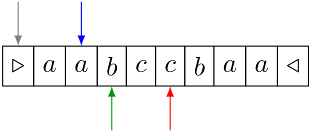

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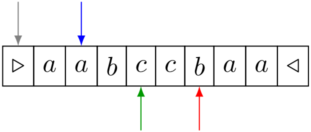

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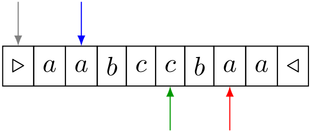

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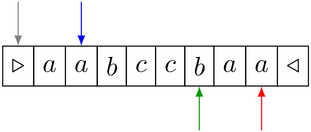

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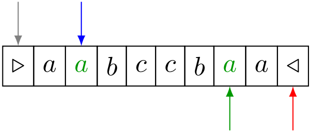

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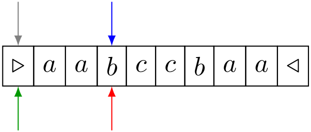
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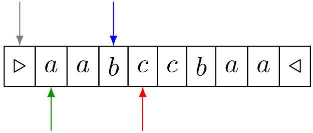

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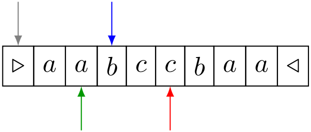

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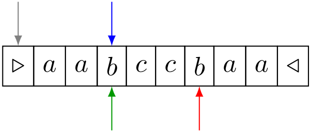

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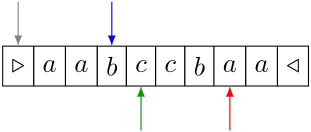
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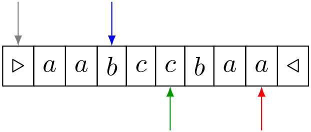

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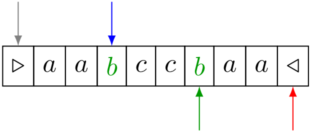

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Generalization of this idea $\Rightarrow$ Translation from 2MA to JMA.

1 Context

2 Computing languages

3 Computing functions

Cut rule and integer functions

The cut rule:

$$\frac{E \vdash e \qquad e, F \vdash g}{E, F \vdash g}$$

- Corresponds to composition of programs.
- Fundamental in proof theory.
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Expressions (simplified): $e, f := 1 \mid e \cdot f \mid e + f \mid e^* \mid e \to f$.

Sequents: $(1^*)^k \vdash 1^*$: functions $\mathbb{N}^k \to \mathbb{N}$.

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Which functions $\mathbb{N}^k \to \mathbb{N}$ can the system with cuts compute ?

System T

As automata before, we want a computational framework to characterize the expressive power of our cyclic proof system.

System T:

- λ-calculus with explicit integer type,
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System T terms:

$$M, N ::= \quad x \mid 0 \mid s(M) \mid \lambda x.M \mid M N \mid \mathbf{Rec}(N, M_0, M_s(x,y))$$

(+constructors/destructors for pairs, lists)

$\mathbf{Rec}(N, M_0, M_s)$ returns $\begin{cases} M_0 & \text{if } N = 0 \\ M_s(N, \mathbf{Rec}(n, M_0, M_s)) & \text{if } N = s(n) \end{cases}$

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Example: Addition $a + b$: $\lambda ab.\mathbf{Rec}(b, a, s(y))$

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System $\mathsf{T_{aff}}$: Affine version of System T, data cannot be duplicated.
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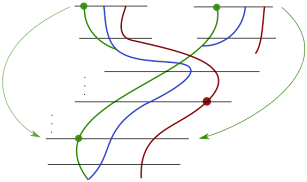

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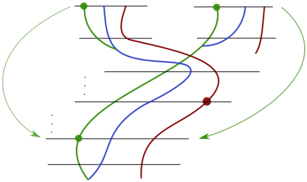

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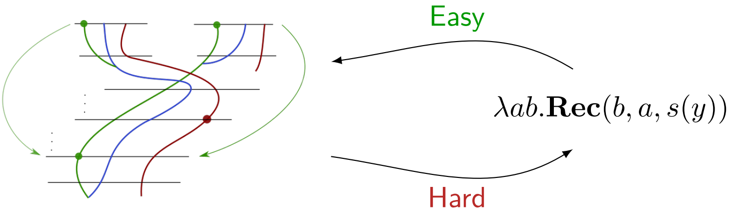

Easy

$\lambda ab.\mathbf{Rec}(b, a, s(y))$

Hard

Results

System $\mathbf{T}_{\mathbf{aff}}$: Affine version of System T, data cannot be duplicated.
Example: $\lambda fx.f(f(x))$ is not typable in $\mathsf{T}_{\mathsf{aff}}$.

Theorem

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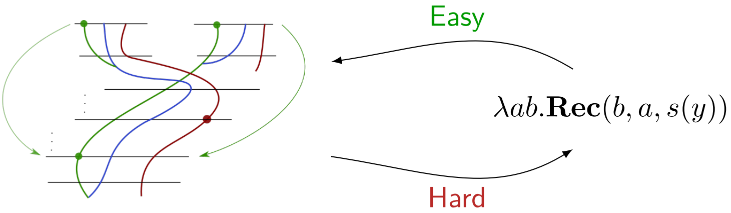

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Hard

Open problems in proof theory: infinite descent versus induction.

Proof schemes

Affine proofs → $\mathsf{T}_{\mathsf{aff}}$:

- normal form for proofs, with explicit hierarchy of cycles,
- inductively build $\mathsf{T}_{\mathsf{aff}}$ terms.

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Proofs → System T:

- ACA_0: $\mathsf{RCA}_0 +$ König's lemma,
- $\forall$ cyclic proof , prove in ACA_0 that its computation terminates,
- Conservativity result: $\mathsf{ACA}_0 \leftrightarrow$ Peano for integer functions,
- Classic result: Peano ↔ System T.

Conclusion

Open problems:

- Avoid the “blackbox” of reverse maths.
- Lift results to transductions.
- Generalize the normal form with hierarchy of cycles to other cyclic proof systems.
- Include greatest fixed point (ω-regular expressions)
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Thank you for your attention !