Good-for-Games automata

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Journées SDA2, Saint-Etienne
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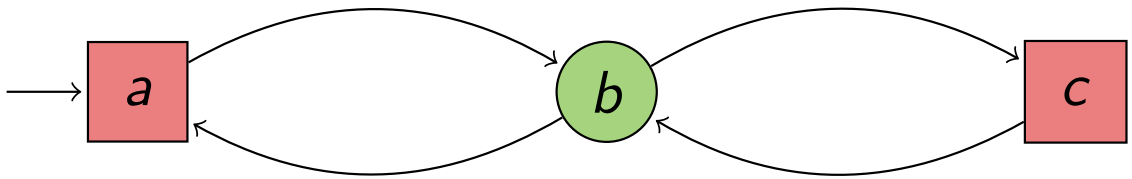
Games

Two Players:

Games

Two Players: Eve Adam

Arena: finite graph $G = (V, E)$, with $V = V_{\bigcirc} \uplus V_{\square}$.



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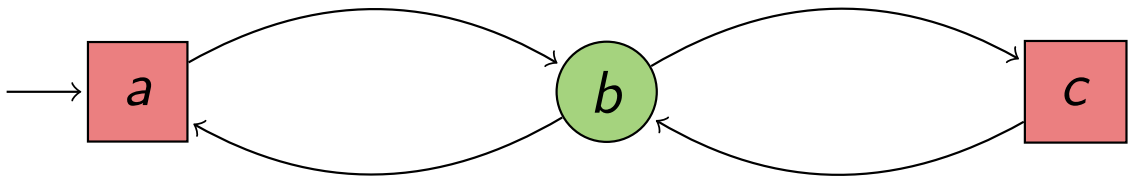

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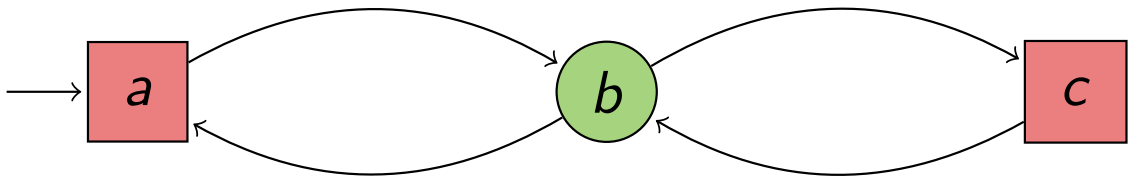

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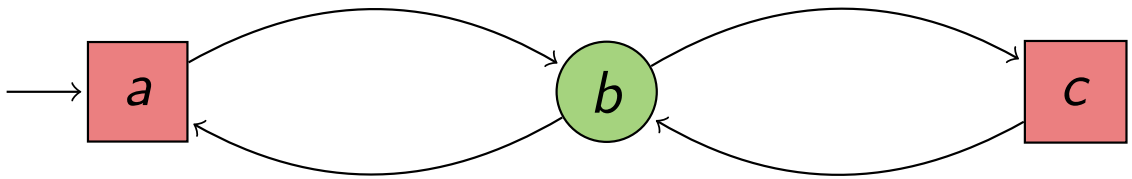

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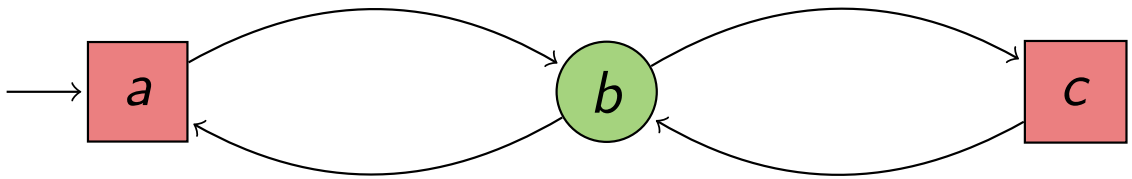

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Eve wins a play π if $\pi \in W$

Strategies

Strategy for Eve: $\sigma_E : V^* V_{\bigcirc} \to V$,
$\sigma_E(v_0 v_1 \dots v_n)$ chooses what to play if $v_0 v_1 \dots v_n$ is the history of the play seen so far..

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Two strategies σ_E, σ_A yield a unique play $\pi(\sigma_E, \sigma_A) = v_0 v_1 v_2 \dots$

- v_0 is the initial vertex,
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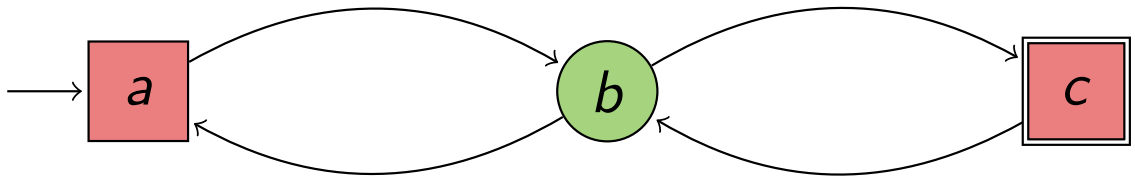
Positional strategy: $\sigma_X : V_X \to V$, for $X \in \{E, A\}$.

Winning conditions

Reachability Condition

Goal for Eve: reach a subset $F \subseteq V$. I.e. $W = V^* F V^\omega$.

Example with $F = \{c\}$:



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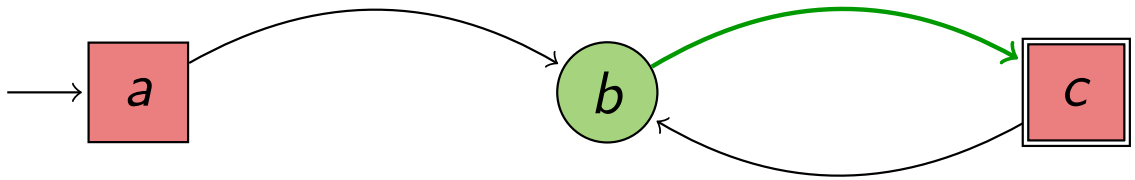

Here: Eve wins with positional strategy.

Winning conditions

Safety Condition

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Dual of Reachability

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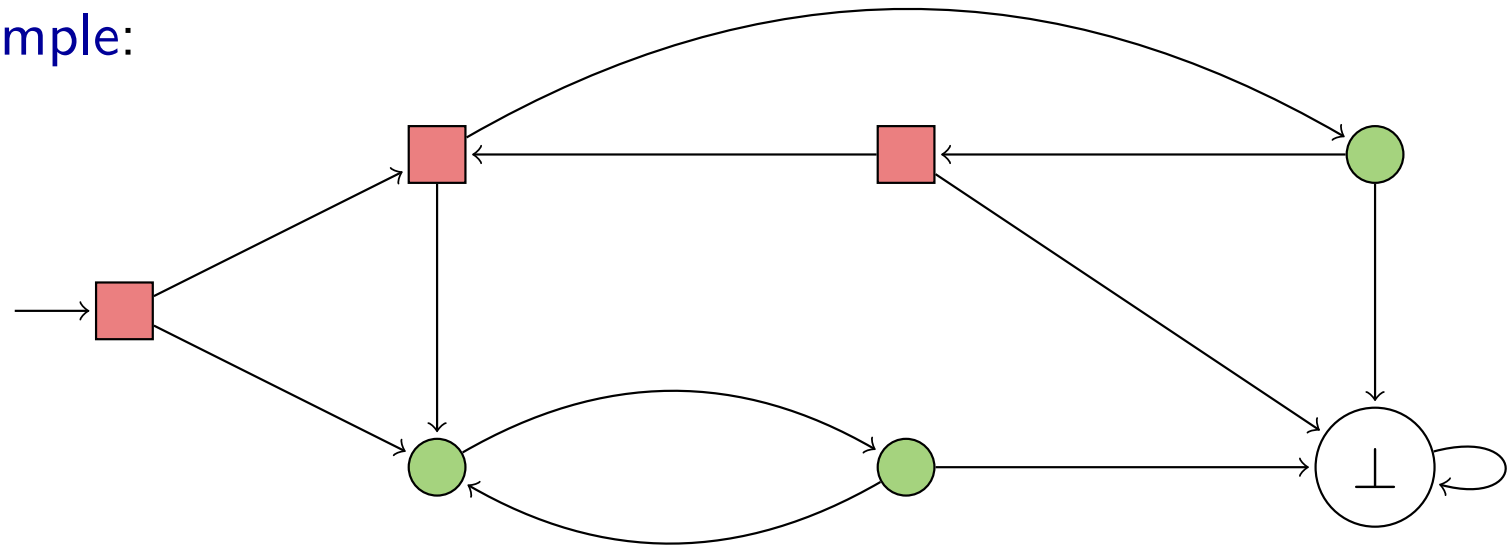

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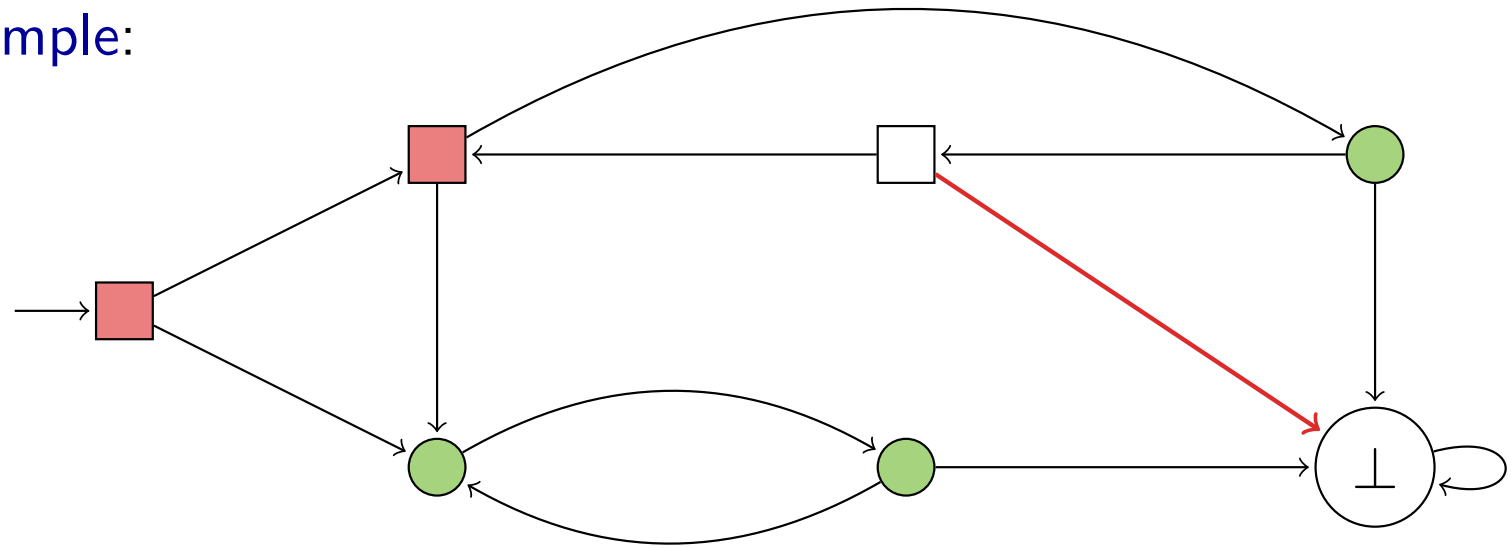

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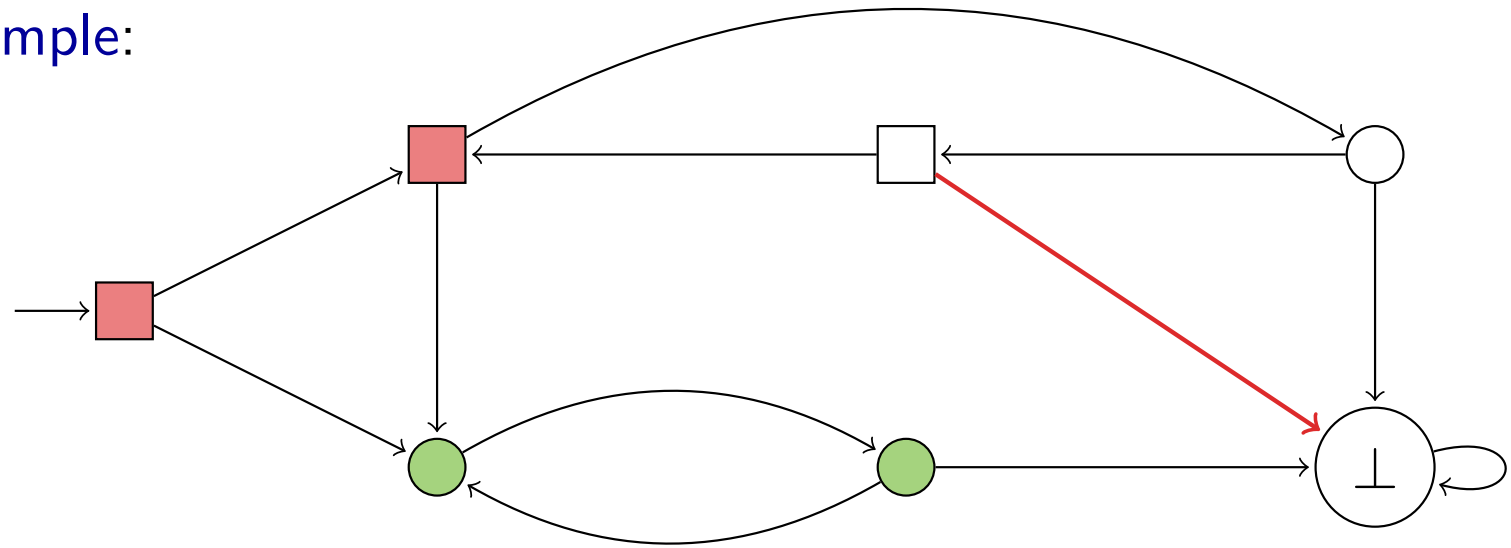

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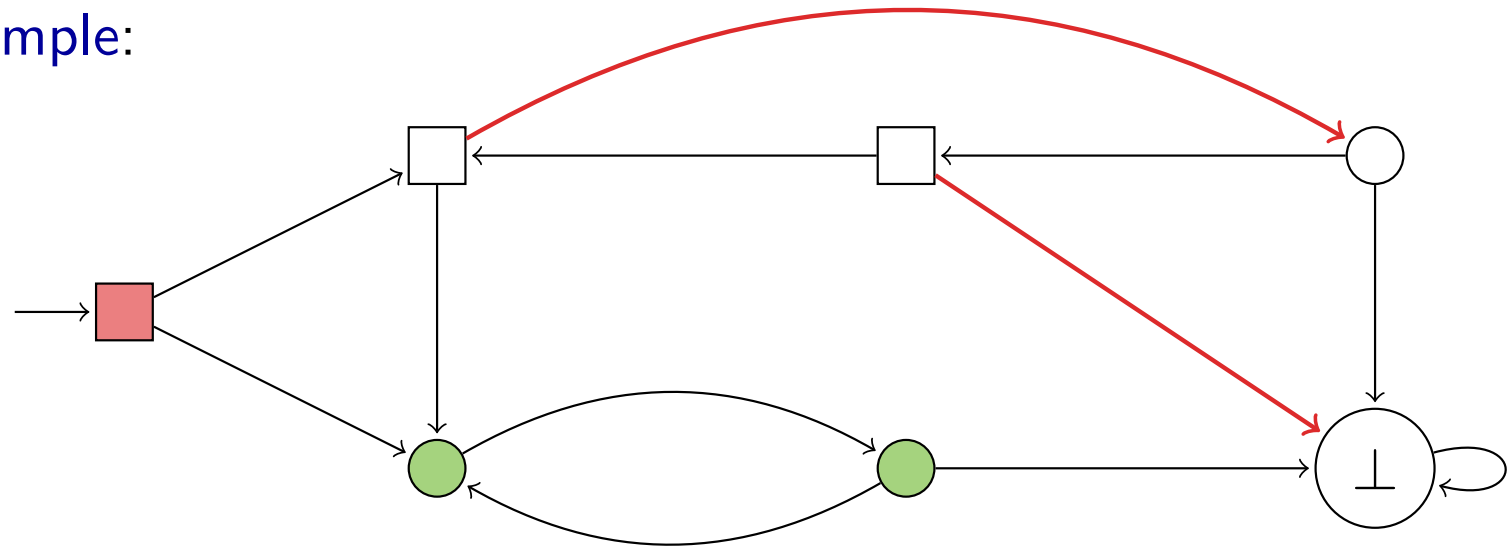

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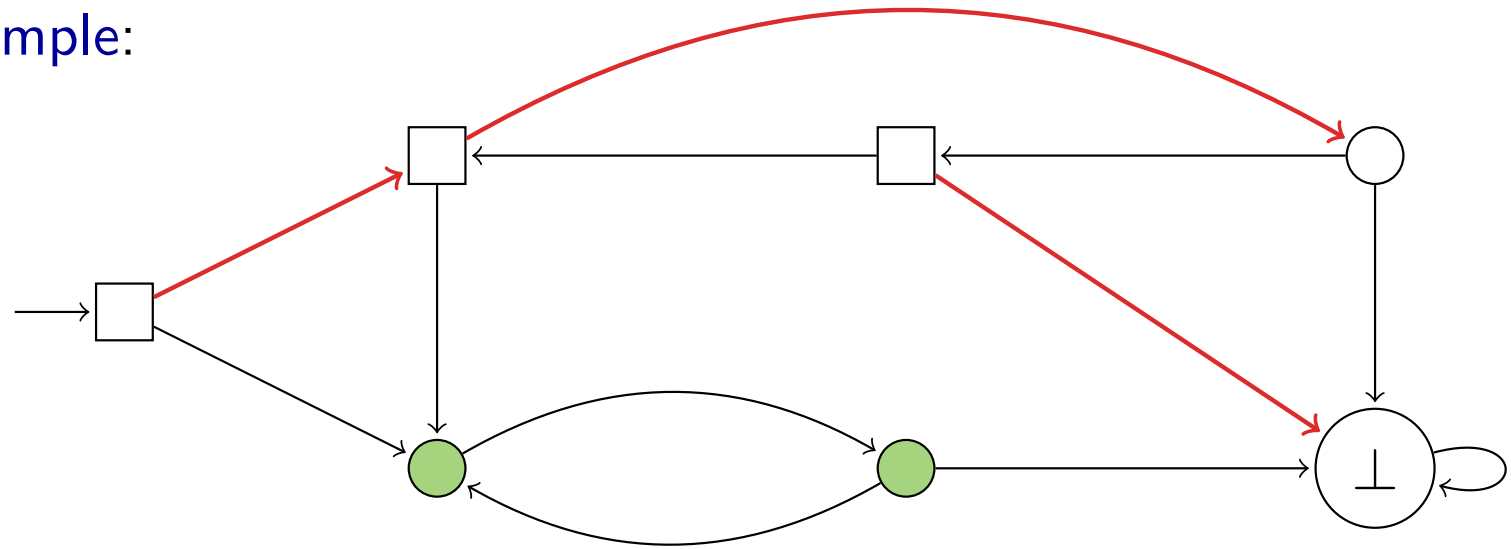

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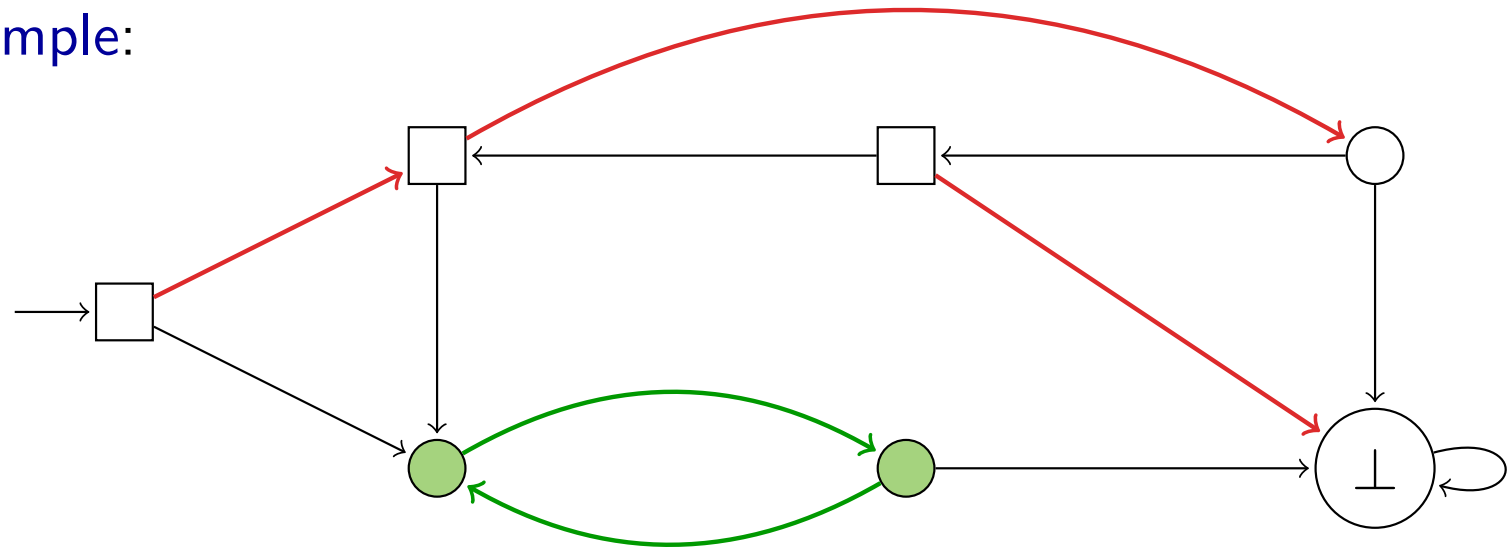

$\rightarrow$ Algorithm for solving Reachability/Safety Games in P.

Corollary: Reachability and Safety games are positionally determined.

Arbitrary conditions

What about arbitrary $W \subseteq V^\omega$? Are they determined ?
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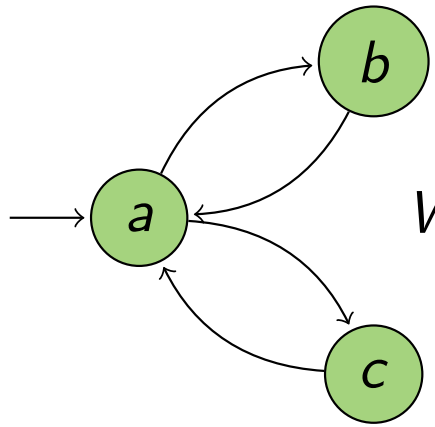

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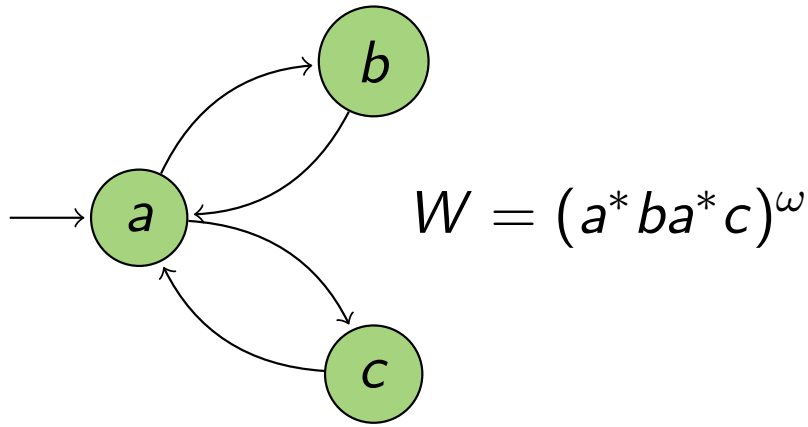

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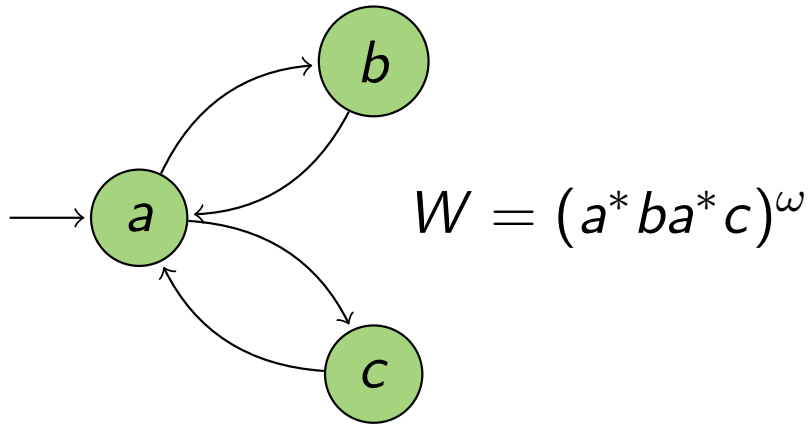

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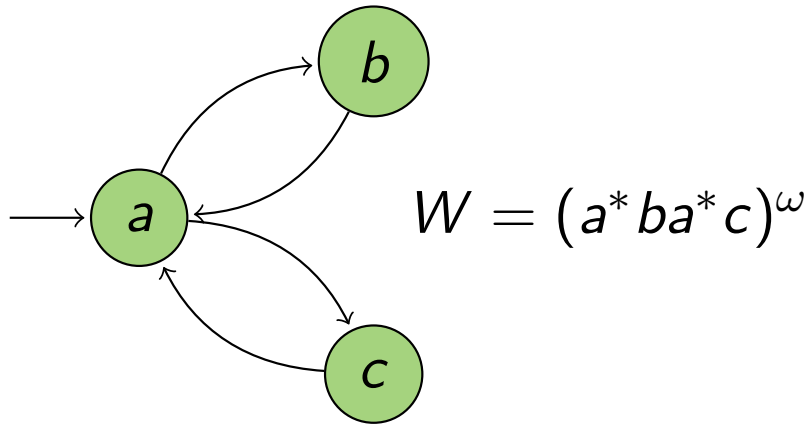

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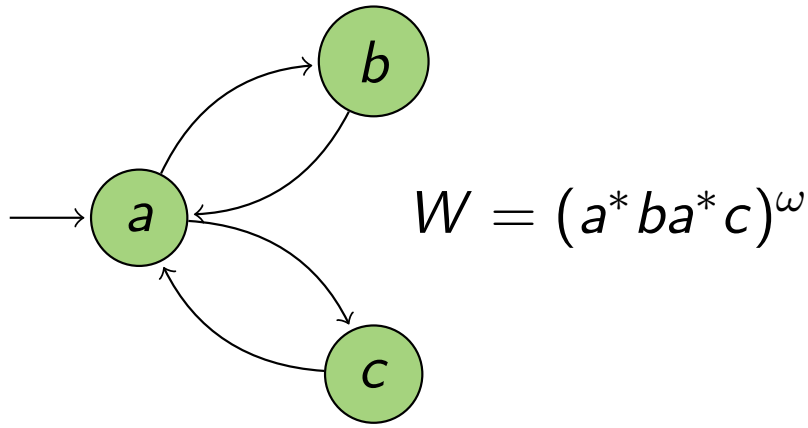

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Theorem

*The **Axiom of determinacy** is uncompatible with ZFC.*
In ZF+DC, it is equivalent to some large cardinals axioms.

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- $[i,j]$-Parity condition: $\mathrm{rank} : V \to \{i, i+1, \ldots, j\}$
  $v_0 v_1 v_2 \cdots \in W$ iff "maximal rank appearing ∞ often is even"
  $\max\{r \mid \exists\infty \text{ many } i \text{ such that } \mathrm{rank}(v_i) = r\}$ is even.

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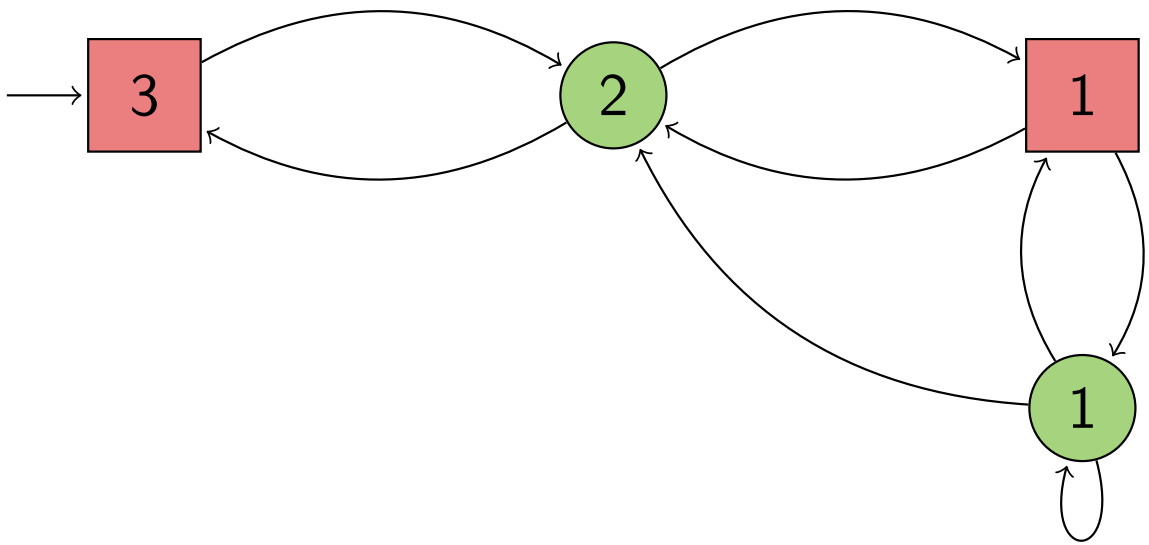

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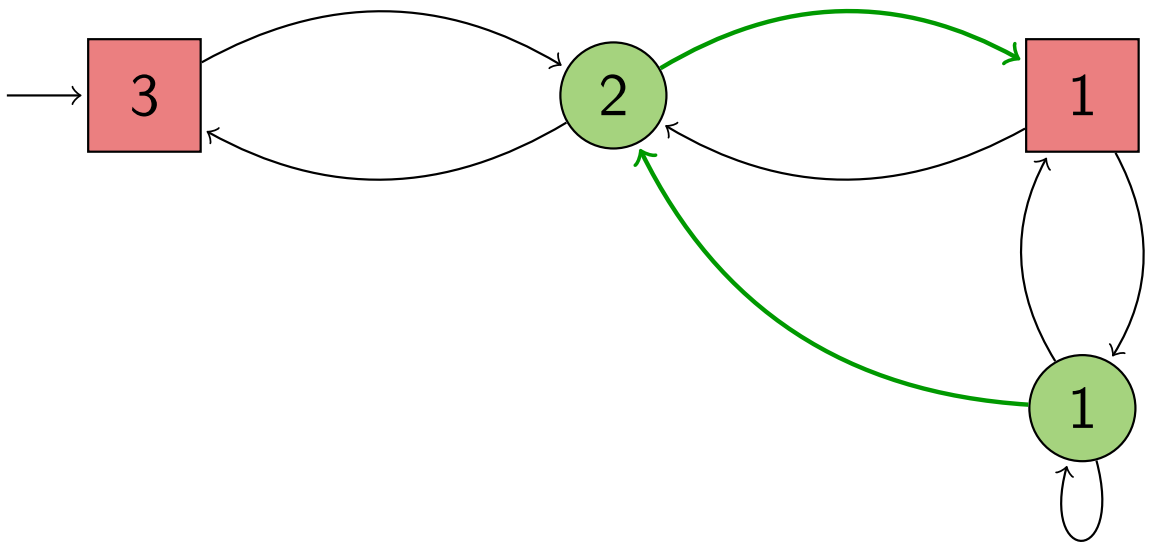

Here: Eve wins positionally.

Results on parity games

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Theorem (Calude, Jain, Khoussainov, Li, Stephan 2017)

Parity games can be solved in Quasi-Polynomial time: $O(n^{\log n+6})$.
For any fixed $[i,j]$, they can be solved in $O(n^5)$.

ω-regular languages

Parity automata on infinite words:

- can be deterministic (Det) or non-deterministic (ND)
- a run is a sequence of state $q_1 q_2 \cdots \in Q^\omega$
- labelling function $\mathrm{rk} : Q \to [i, j]$
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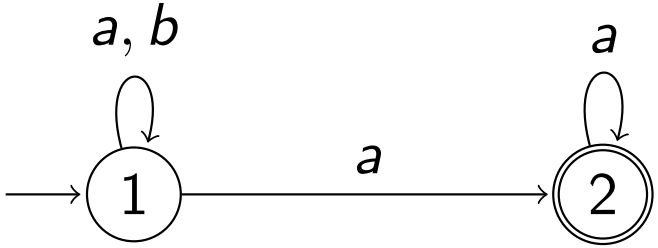

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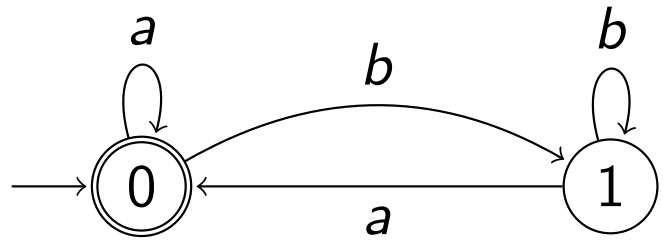

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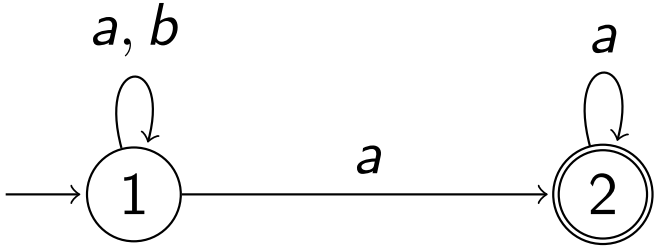

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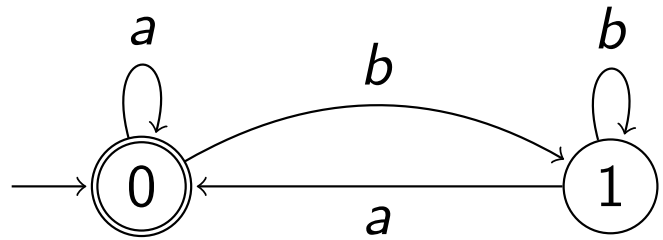

Det coBüchi

Theorem (McNaughton 1966)

ND Büchi $\Leftrightarrow$ *Det Parity* $\Leftrightarrow$ *ND Parity* ($\Leftrightarrow \ldots$)

Languages as winning conditions

What if the winning W condition of G can be any ω-regular language ?

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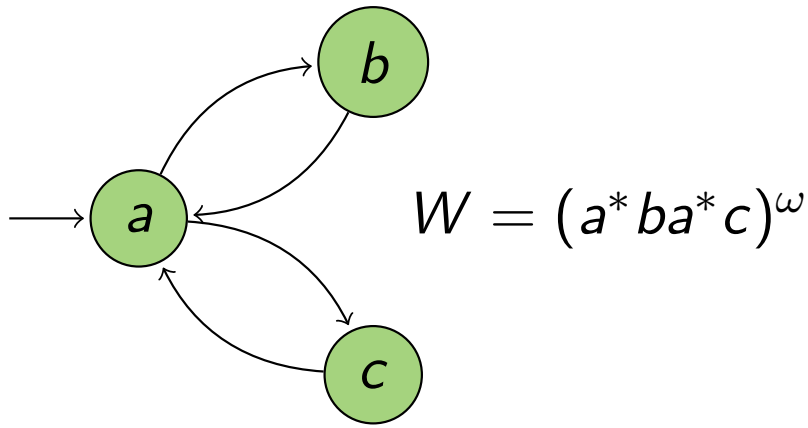

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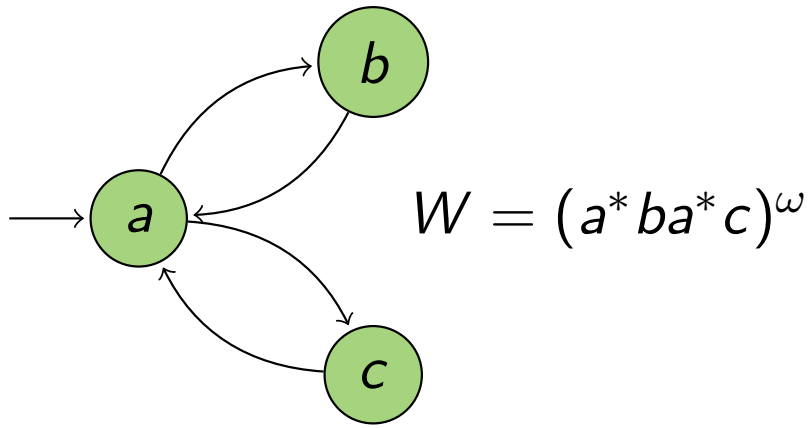

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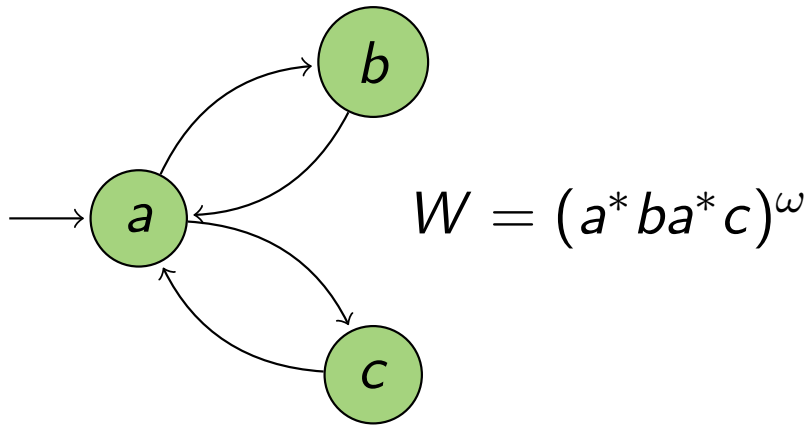

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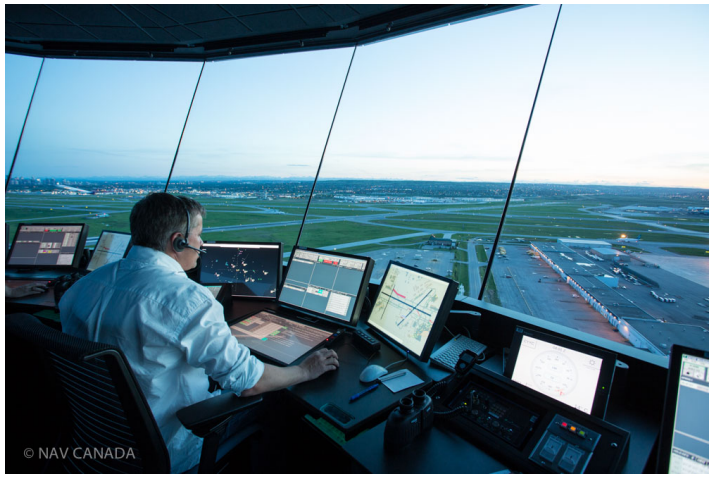
Applications in Logic, Verification, Proof theory, Complexity, ...

Church's synthesis problem (1962)



$\longleftarrow I_1$

$\Longrightarrow O_1$

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Desired algorithm: Specification $\varphi \rightsquigarrow$ Controller C
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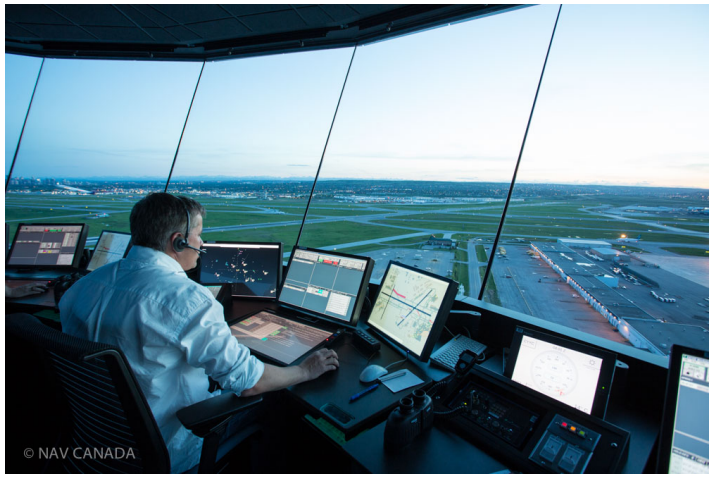

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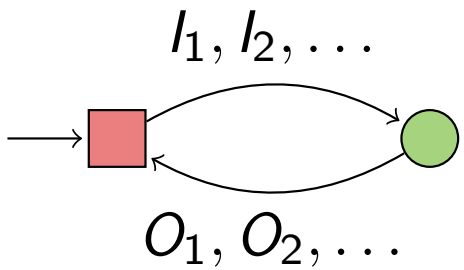

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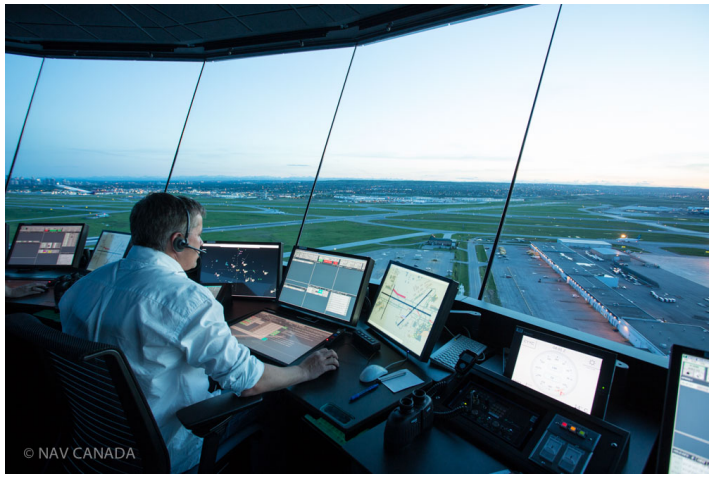

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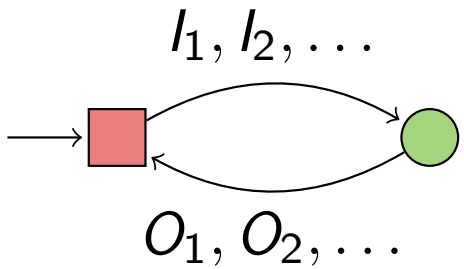

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What is $G \circ \mathcal{A}_{\mathrm{Det}}$?

- States: $V' = V \times Q_{\mathrm{Det}}$,
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Theorem
$G \circ \mathcal{A}_{\mathrm{Det}}$ has same winner as G.

Do we need to determinize ?

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When $\mathcal{A}$ is deterministic, $G \circ \mathcal{A}$ has same winner as G.

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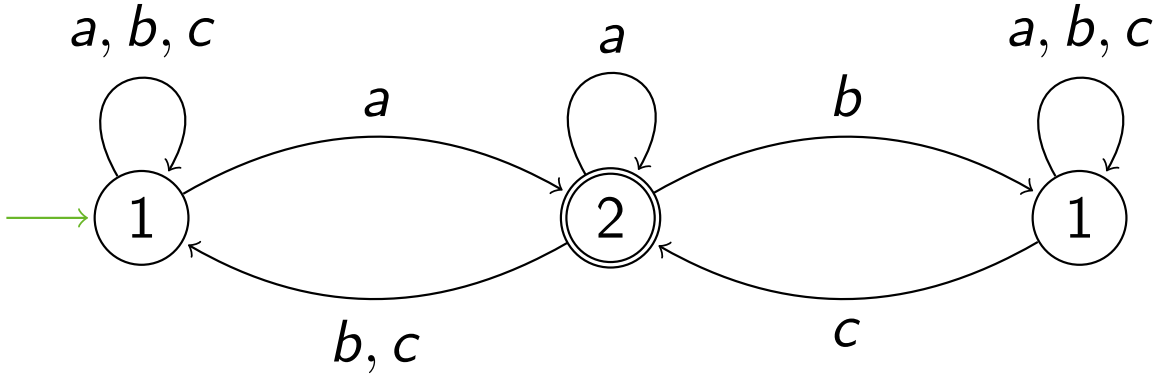
$\mathcal{A}$ Parity automaton, we associate to it a **GFG game**:
Adam plays letters:

Eve: resolves non-deterministic choices for transitions



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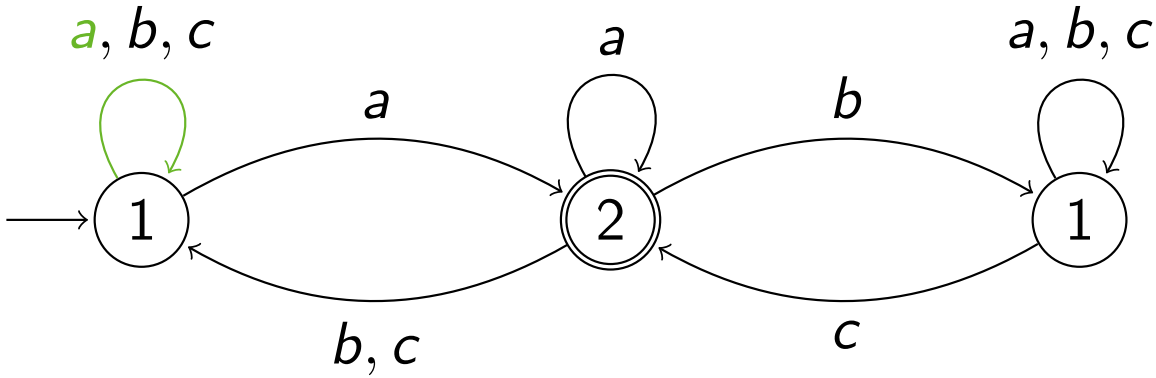

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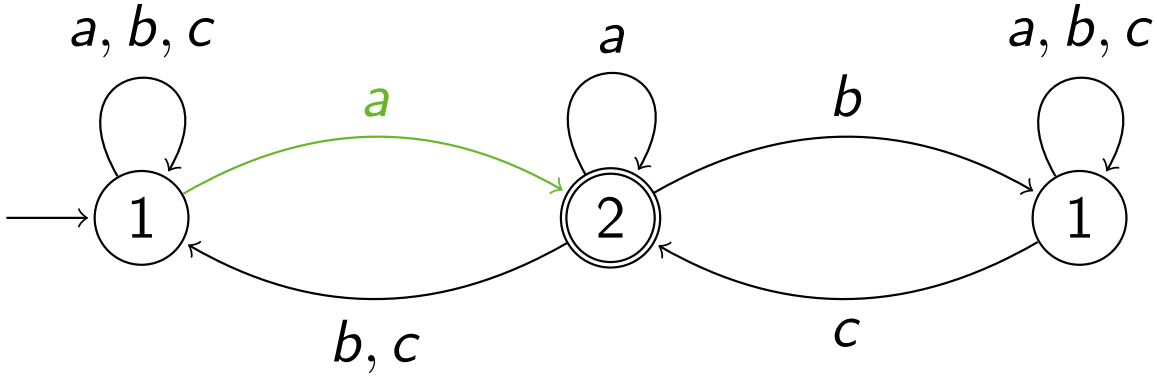

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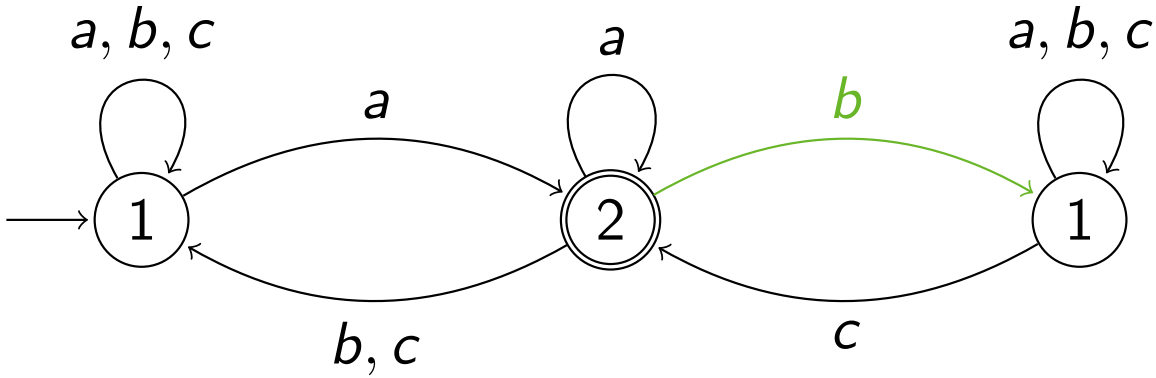

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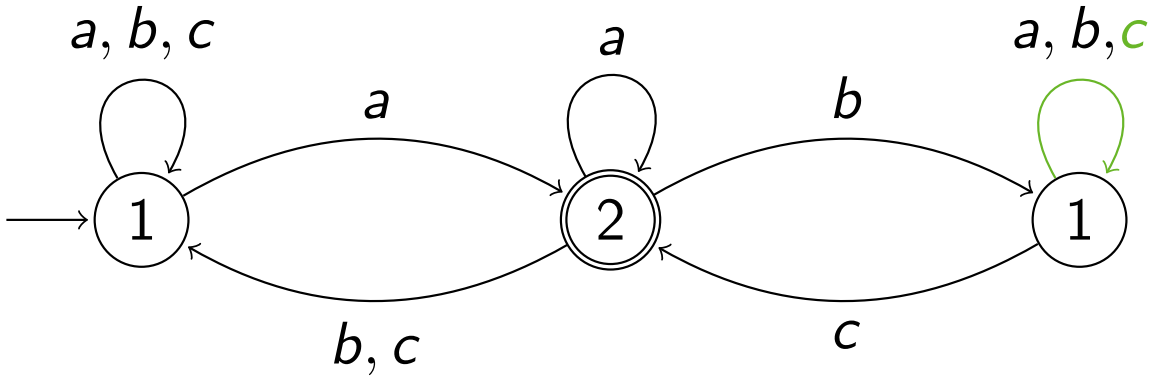

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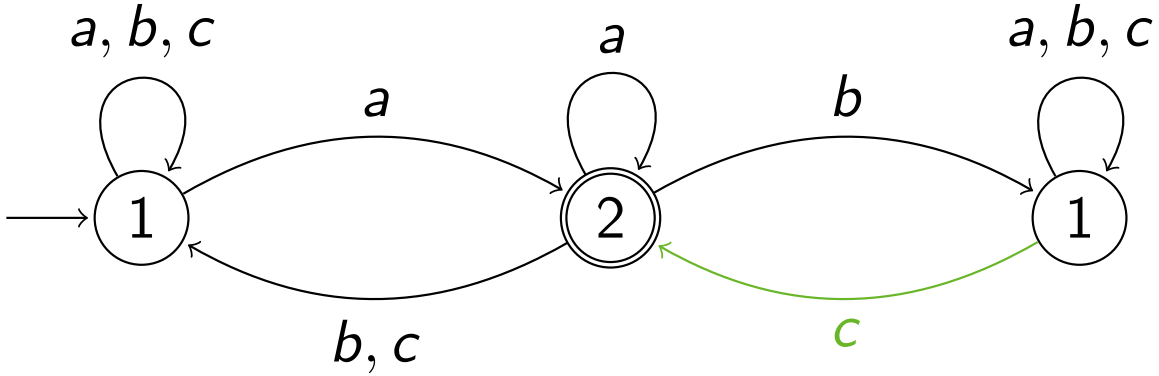

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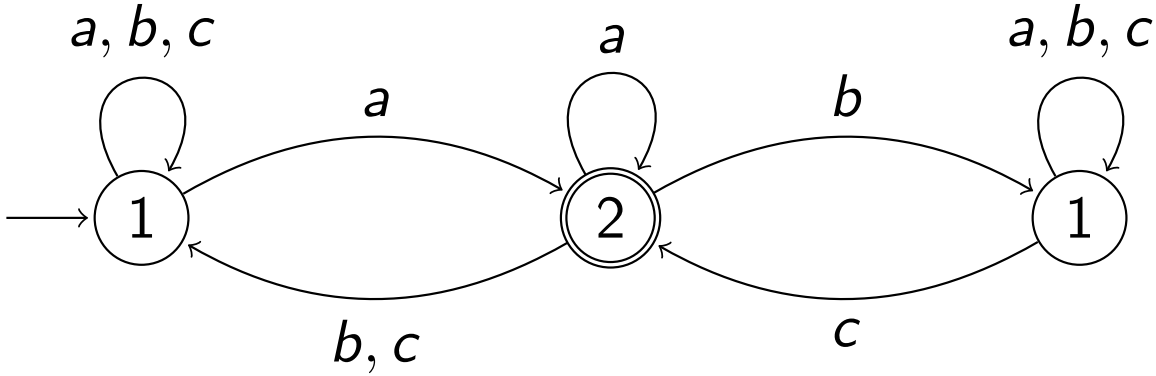

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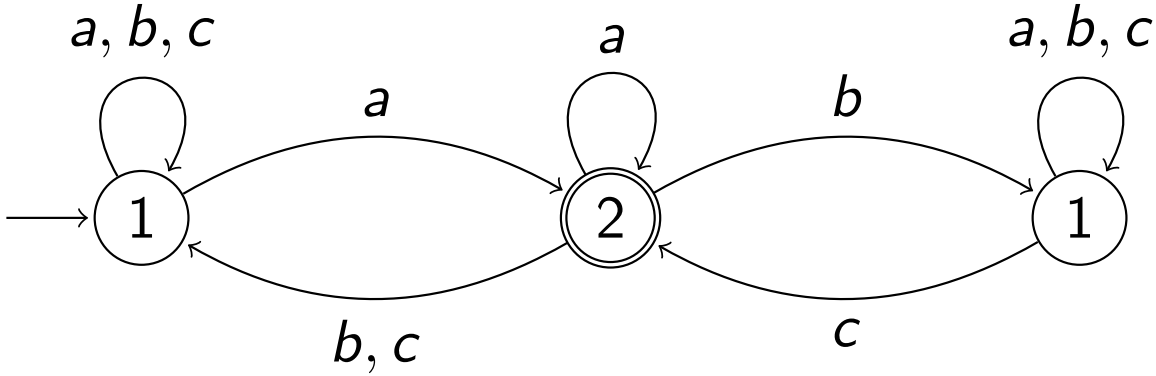

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$\mathcal{A}$ **GFG** means that there is a strategy $\sigma_{\mathrm{GFG}} : A^* \to Q$, for accepting all words of $L(\mathcal{A})$.

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Every deterministic automaton is **GFG**.

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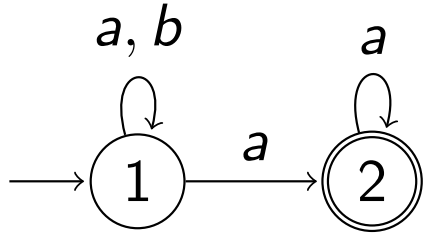

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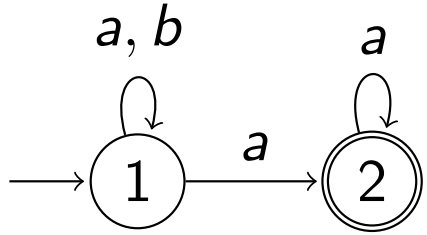

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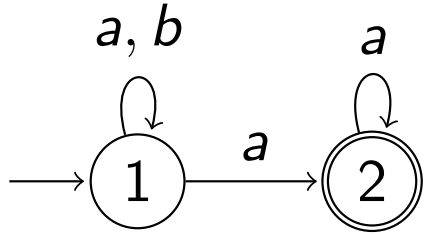

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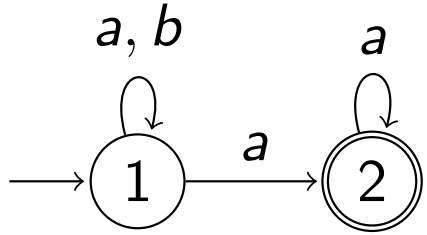

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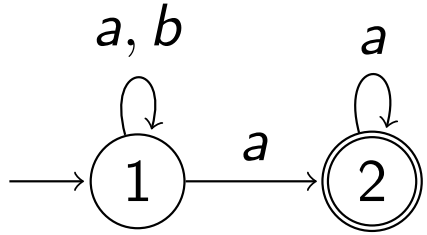

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$\rightarrow$ we can ignore the strategy in algorithms like synthesis.

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Lemma: For Reachability/Safety conditions, **GFG** $\approx$ Det.

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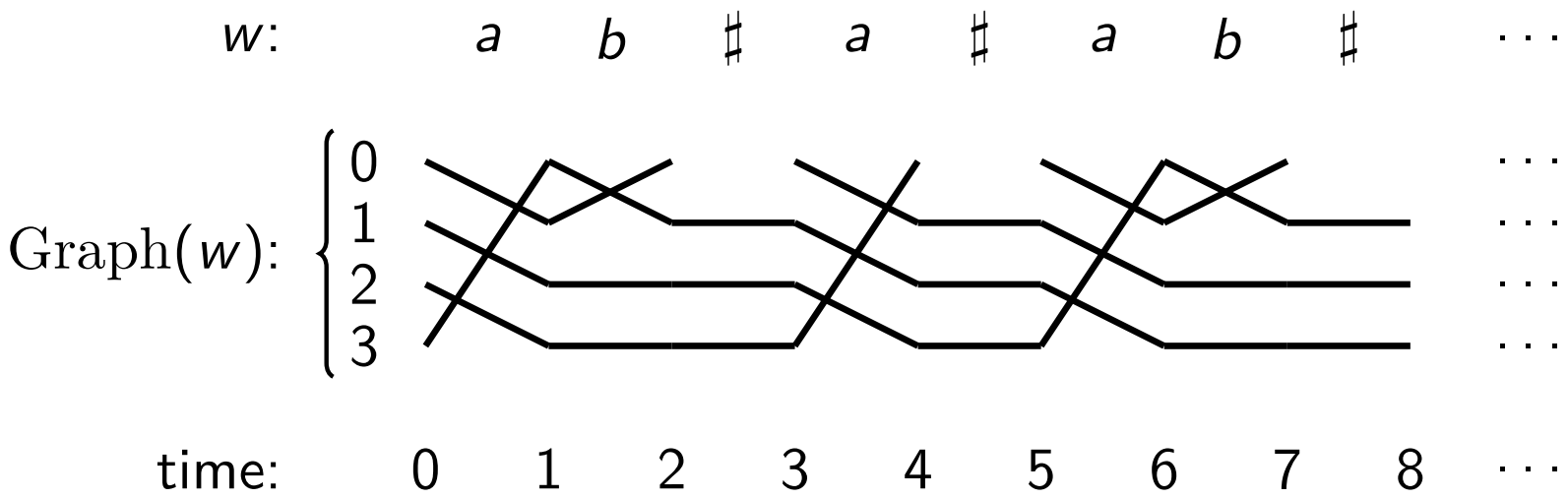

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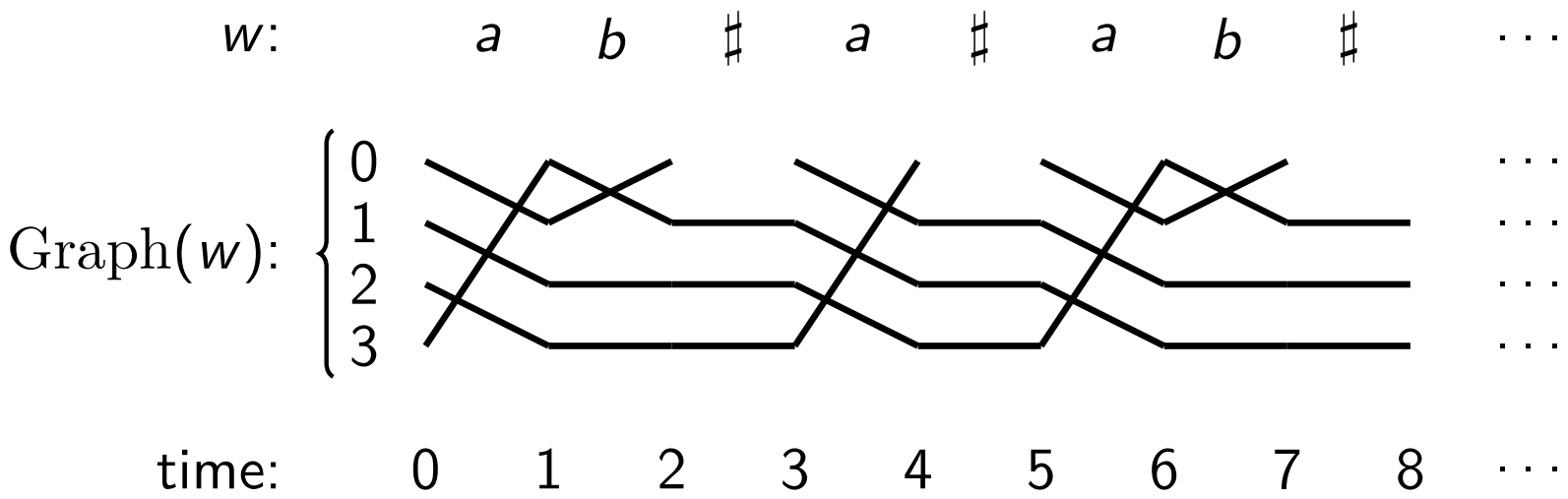

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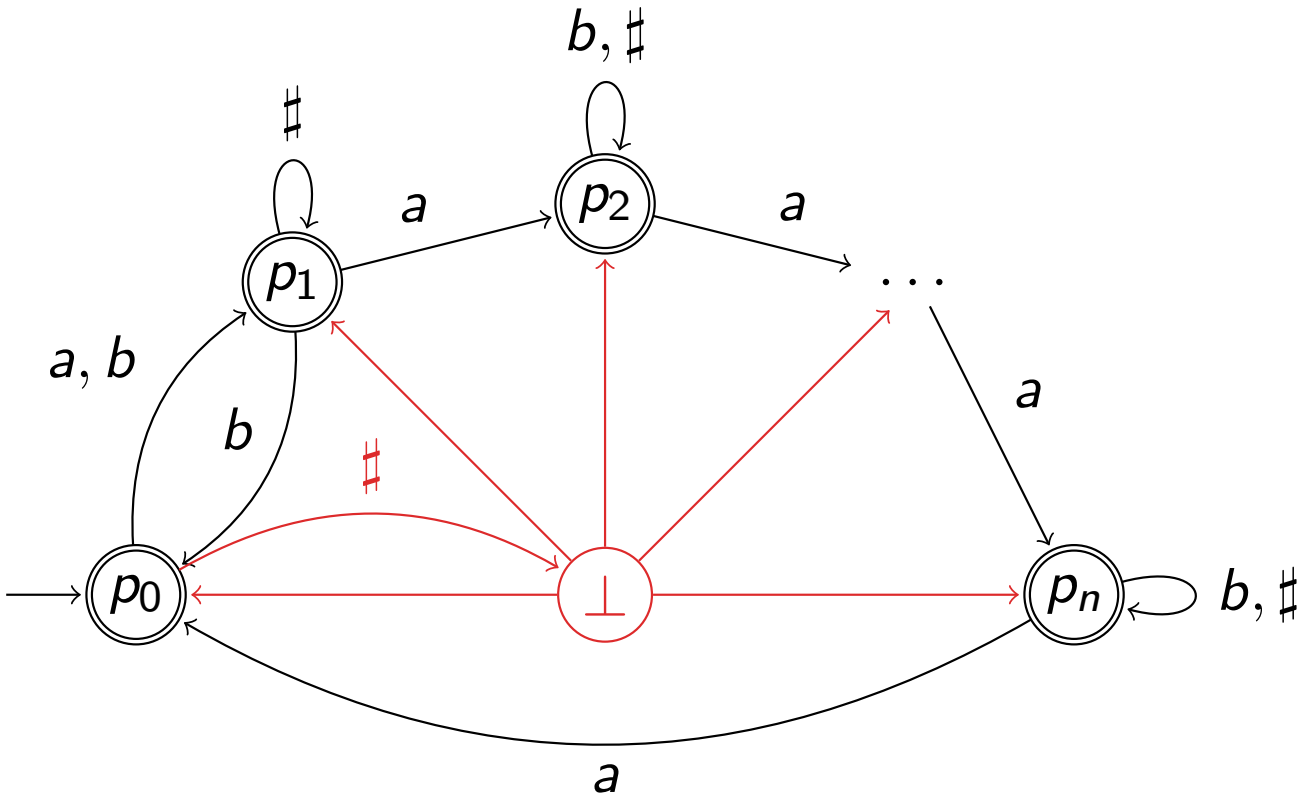
Any Det Parity automaton for L_n needs $\dfrac{2^{\frac{n}{2}}}{n+1}$ states !

A small GFG automaton for L_n

GFG coBüchi automaton:



Idea: Try paths one after the other.

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Conjecture: G_2 is equivalent to GFG for all parity automata.

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Directly from formula φ [Iosti, K. (unpublished)]
Tailored for **Synthesis Problem**
For now: Fragment of coBüchi languages

Conclusion

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Open problems

- Complexity of **GFGness problem** (even for 3 parity ranks).
- New ways to build **GFG** automata.
- Generalization to other models (alternating, pushdown, . . .).