Etude de classes de fonctions de coût régulières

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Soutenance de thèse
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Introduction

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  Which problems can be answered by an algorithm ?
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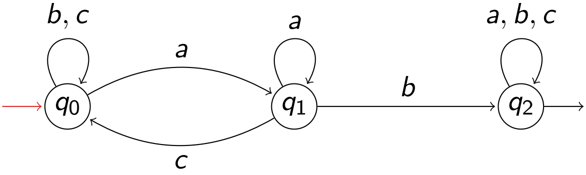
- **Finite Automata:** Formalism with decidable properties.
- **Automata theory:**
  Toolbox to decide many problems arising naturally.
  Verification of systems can be done automatically.
  Theoretical and practical advantages.

1. Automata theory
2. Regular Cost Functions
3. Contributions of the thesis
4. Zoom: Aperiodic Cost Functions

Finite Automaton



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Example : Accepting path for the word $babc$.

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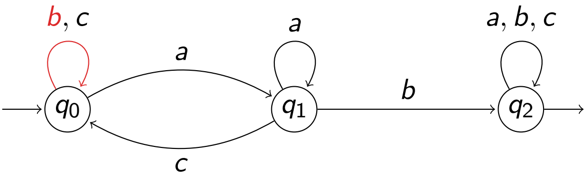

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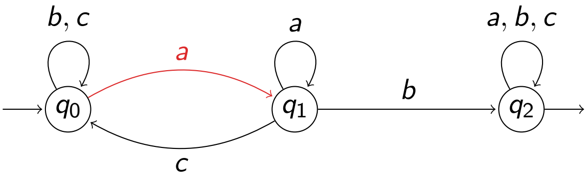

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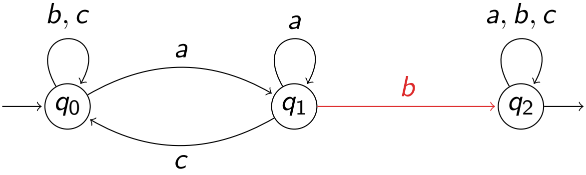

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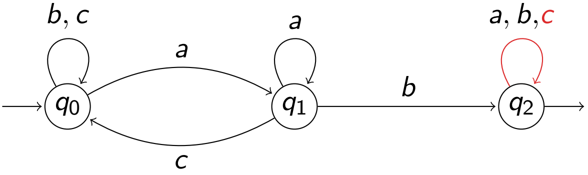

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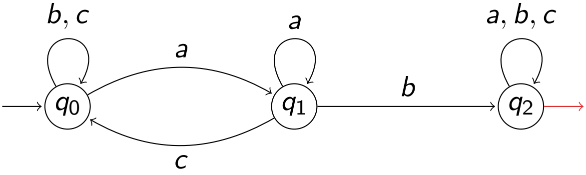

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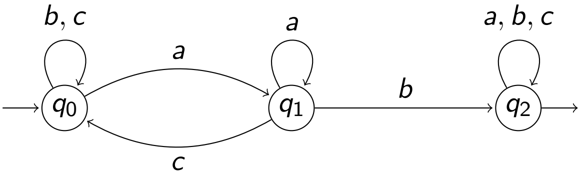
The language recognized by $\mathcal{A}$ is the set $L \subseteq \mathbb{A}^*$ of words accepted by $\mathcal{A}$.

Descriptions of a language



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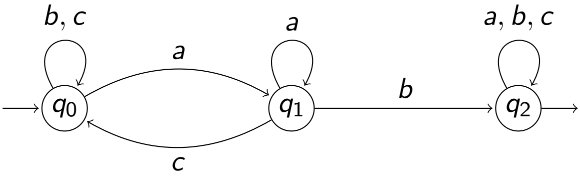

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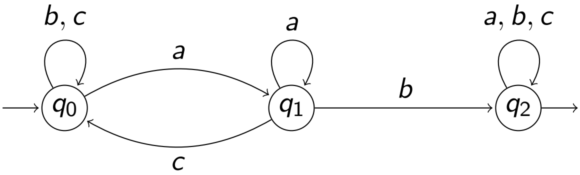

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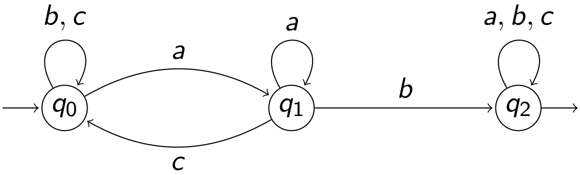

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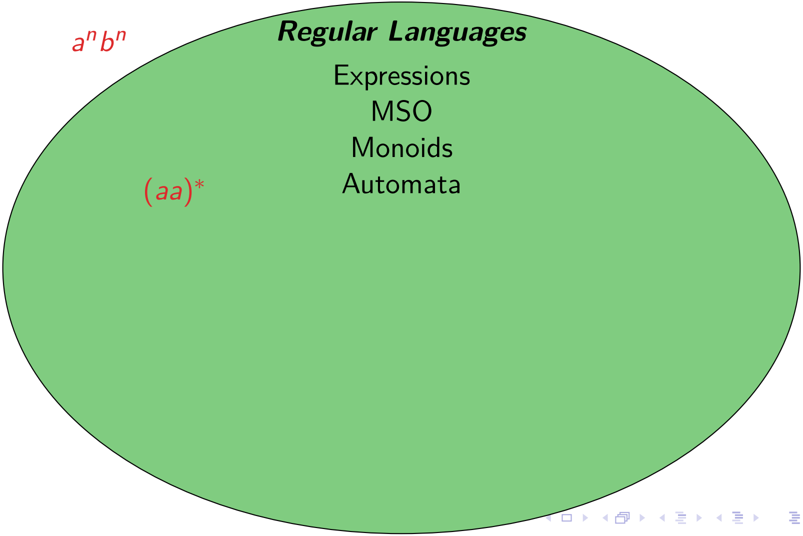
Are there always translations from one formalism to another ?

Regular Languages

All these formalisms are effectively equivalent.



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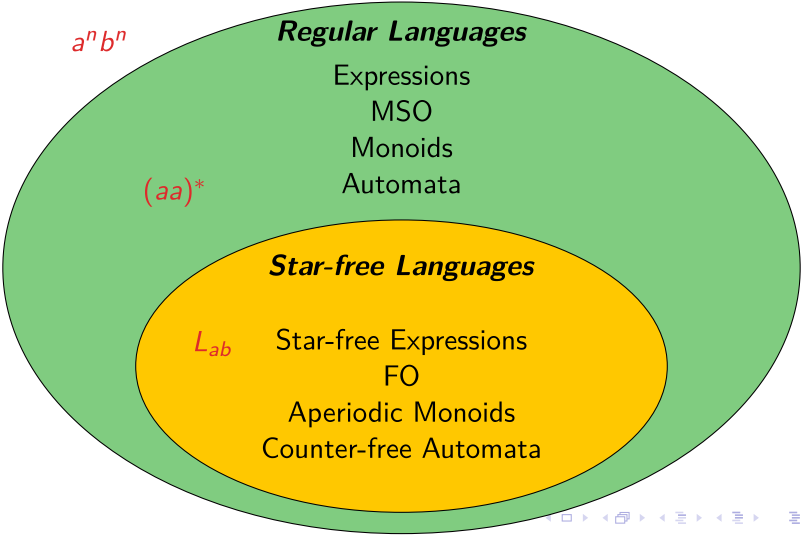

Historical motivation

Given a class $\mathcal{C}$, is there an algorithm deciding $L \in \mathcal{C}$?

Theorem (Schützenberger 1965)

It is decidable whether a regular language is star-free, thanks to the equivalence with aperiodic monoids.

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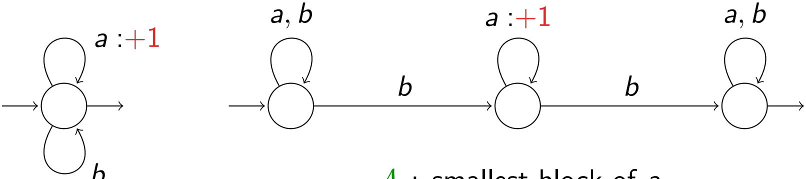
Finite Power Problem: Given L, is there n such that

$$(L + \varepsilon)^n = L^* \ ?$$

There is no known algebraic characterization,
other technics are needed to show decidability.

Cost Automata



$\mathcal{A}_1$: number of a

$\mathcal{A}_2$: smallest block of a

Unbounded: There are words with arbitrarily large value.

Cost Automata: More counters, reset operations,. . .

Deciding **Boundedness** for cost automata$\Rightarrow$ finite power problem.

Theorem (Hashiguchi 82, Kirsten 05)

Boundedness is decidable for cost automata.

Problems solved using counters

- ***Finite Power*** (finite words) [Simon '78, Hashiguchi '79]
  Is there n such that $(L+\varepsilon)^n = L^*$?
- ***Fixed Point Iteration*** (finite words)
  [Blumensath+Otto+Weyer '09]
  Can we bound the number of fixpoint iterations in a MSO formula ?
- ***Star-Height*** (finite words/trees)
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]
  Given n, is there an expression for L, with at most n nesting of Kleene stars?
- ***Parity Rank*** (infinite trees)
  [reduction in Colcombet+Löding '08, decidability open, deterministic input Niwinski+Walukiewicz '05]
  Given $i < j$, is there a parity automaton for L using ranks $\{i, i+1, \ldots, j\}$?

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Theory of Regular Cost Functions

Aim: General framework for previous constructions.

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Theorem (Colcombet '09, following Hashiguchi, Leung, Simon, Kirsten, Bojańczyk+Colcombet)

Cost automata $\Leftrightarrow$ *Cost logics* $\Leftrightarrow$ *Stabilisation monoids.*
Boundedness is decidable.

It provides a toolbox to decide boundedness problems.

Languages as cost functions

A language L is represented by its characteristic function

$$\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$$

Cost function theory strictly extends language theory.

All theorems on cost functions are in particular true for languages.

Goal of the thesis: Studying cost function theory, and generalise known theorems from languages to cost functions.

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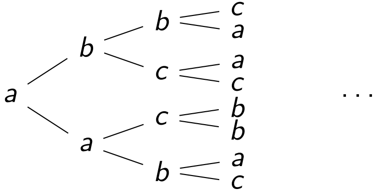
Contributions of the thesis

Input structures:

Finite words: *accba*

Infinite words: *abaabaccbaba* . . .

Infinite trees:



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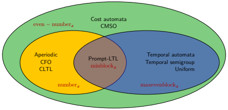

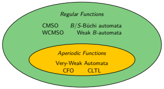

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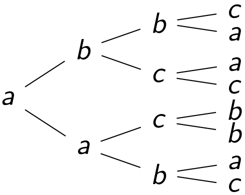

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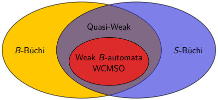
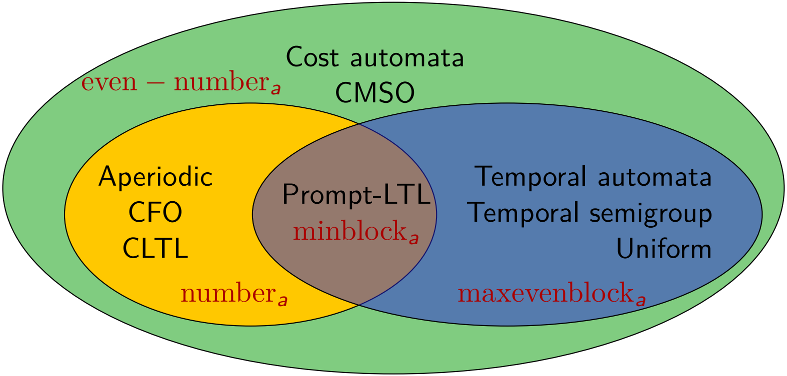


Different kinds of results:

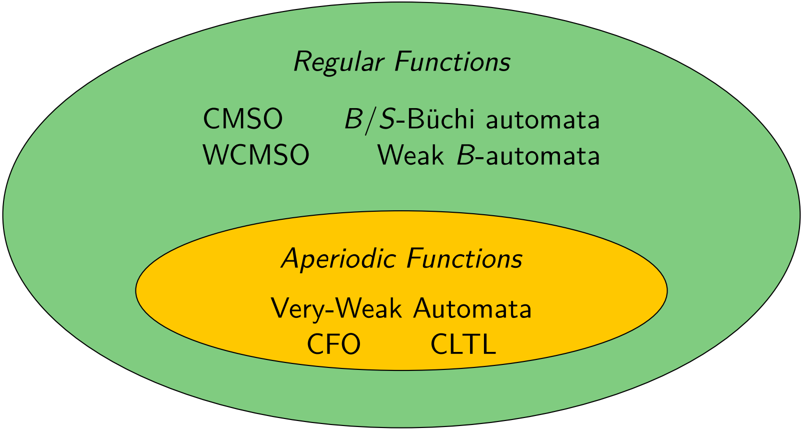
- Generalisation of language notions and theorems,
- Study of classes specific to cost functions,
- Reduction of classical decision problems to boundedness problems.

Cost Functions on finite words



Decidability of membership and effectiveness of translations [Colcombet+K.+Lombardy ICALP '10, K. STACS '11].
Generalization of Myhill-Nerode Equivalence [K. STACS '11].

Cost Functions on infinite words



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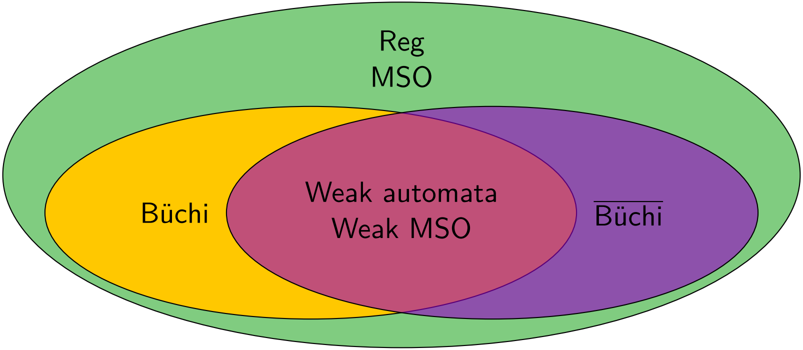
Languages on infinite trees

Theorem (Rabin 1970, Kupferman + Vardi 1999)

L recognizable by an alternating weak automaton $\Leftrightarrow$
L recognizable by WMSO $\Leftrightarrow$ there are Büchi automata $\mathcal{U}$ and $\mathcal{U}'$ such that $L = L(\mathcal{U}) = \overline{L(\mathcal{U}')}$.



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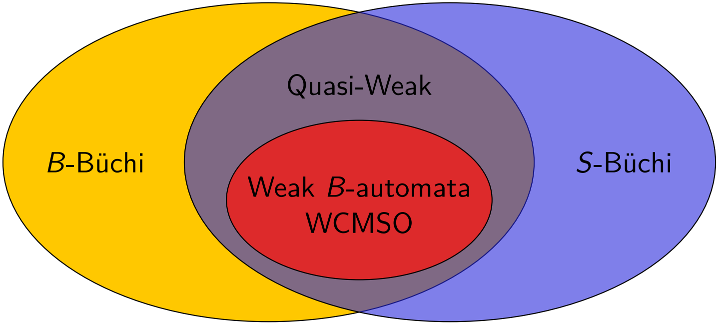

Theorem (K.+Vanden Boom, FSTTCS 2011)

- *B-Büchi $\cap$ S-Büchi = Quasi-Weak $\supsetneq$ Weak.*
- *Decidability of boundedness for Quasi-Weak automata.*
- *If $\mathcal{A}$ is a Büchi automaton, it is decidable whether $L(\mathcal{A})$ is weak [unpublished].*

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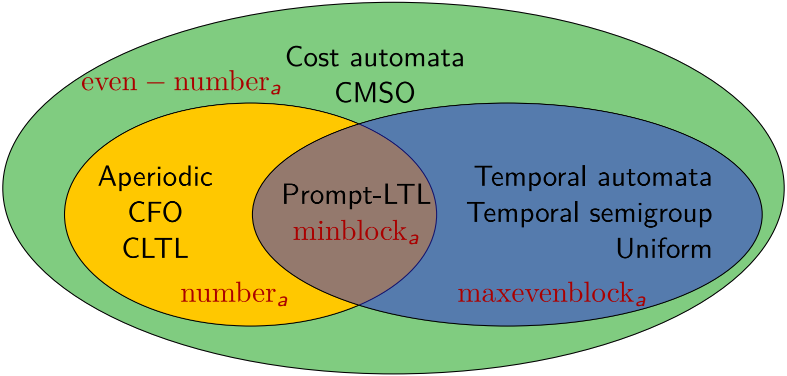
Cost Functions on finite words



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Logics on Finite Words

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- MSO: FO with quantification on sets, noted X, Y.
- Linear Temporal Logic (LTL) over $\mathbb{A}^*$:

$$\varphi := a \mid \Omega \mid \neg\varphi \mid \varphi \vee \psi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \psi$$

$\varphi \mathbf{U} \psi$: $\quad$ $\varphi\ \varphi\ \varphi\ \varphi\ \varphi\ \varphi\ \varphi\ \varphi\ \psi$ over $a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9 a_{10}$

Future operators **G** (Always) and **F** (Eventually).
Example: To describe L_{ab}, we can write $\mathbf{F}(a \wedge \mathbf{X}b)$.

Generalisation: cost LTL

- CLTL over $\mathbb{A}^*$:

$$\varphi := a \mid \Omega \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \psi \mid \varphi \mathbf{U}^{\leq N} \psi$$

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$\varphi \mathbf{U}^{\leq N} \psi$:

φ	φ	$\times$	φ	φ	$\times$	φ	φ	ψ		
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}

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- "Error variable" N is unique, shared by all occurences of $\mathbf{U}^{\leq N}$.
- $\mathbf{G}^{\leq N}\varphi$: φ is false at most N times in the future ($\varphi \mathbf{U}^{\leq N} \Omega$).

Generalisation : Cost FO and Cost MSO

- CFO over $\mathbb{A}^*$:

  $$\varphi := a(x) \mid x = y \mid x < y \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi \mid \forall x \varphi \mid \forall^{\leq N} x \varphi$$

  Negations pushed to the leaves.
- As before, N unique free variable.
- $\forall^{\leq N} x \varphi(x)$ means φ is false on at most N positions.
- CMSO extends CFO by allowing quantification over sets.

Semantics of Cost Logics

From formula to cost function:

Formula $\varphi \longrightarrow$ cost function $[\![\varphi]\!] : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, defined by

$$[\![\varphi]\!](u) = \inf\{n \in \mathbb{N} : \varphi \text{ is true over } u \text{ with } n \text{ as error value}\}$$

Example with the alphabet $\{a, b\}$

- $\mathrm{number}_a = [\![\mathbf{G}^{\leq N} b]\!] = [\![\forall^{\leq N} x \;\; b(x)]\!]$.

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  $= [\![\forall X \ \mathrm{block}_a(X) \Rightarrow (\forall^{\leq N} x \ x \notin X)]\!]$.
- If φ is a classical formula for L, then $[\![\varphi]\!] = \chi_L$.

Stabilisation monoids

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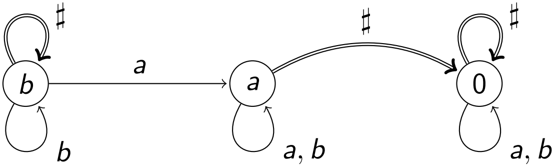
Example: Stabilisation Monoid for number_a

$M = \{b, a, 0\}$, $P = \{a, b\}$,

b**:** "no a", a**:** "a little number of a", 0**:** "a lot of a".



Cayley graph

Aperiodic Monoids

Definition: A [stabilisation] monoid M is aperiodic if for all $x \in M$ there is $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

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Theorem (McNaughton-Papert, Schützenberger, Kamp)

Aperiodic Monoids $\Leftrightarrow$ FO $\Leftrightarrow$ LTL $\Leftrightarrow$ Star-free Expressions.

We want to generalise this theorem to cost functions.
The problems are:

- No complementation $\Rightarrow$ No Star-free expressions.
- Deterministic automata are strictly weaker.
- Heavy formalisms (semantics of stabilisation monoids).
- New quantitative behaviours.
- Original proofs already hard.

Aperiodic cost functions

Theorem (K. STACS 2011)

Aperiodic stabilisation monoid $\Leftrightarrow$ *CLTL* $\Leftrightarrow$ *CFO.*

Proof Ideas:

- Generalisation of Myhill-Nerode $\Rightarrow$ Syntactic object.
- Induction on $(|M|, |\mathbb{A}|)$.
- Extend functions to sequences of words.
- Use bounded approximations.
- Extend CLTL with Past operators, show Separability.

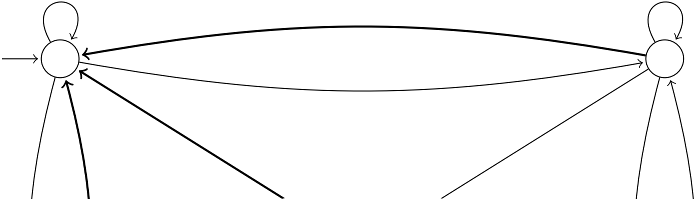
Conclusion

Looking back:

- Exploration of cost function theory.
- Learned a lot about automata theory, proof techniques.
- Lots of results about cost functions, one about languages.
- Some projects did not succeed.

And after ?

- Open problems,
- Works in progress: Quasi-Weak class, tree languages,...
- Applications (verification, language theory).



Merci !