Regular cost functions

Denis Kuperberg
(joint work with Thomas Colcombet and Sylvain Lombardy)
Liafa, équipe Automates

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Outline

Cost functions

Notations : $\mathbb{A}$ finite alphabet, $\omega + 1 = \omega \cup \{\omega\}$
If $f, g : \mathbb{A}^* \to \omega + 1$, then

$$f \approx g \text{ iff } \forall X \subseteq \mathbb{A}^*, f|_X \text{ bounded } \Leftrightarrow g|_X \text{ bounded.}$$

$$\text{iff } \exists\, \alpha : \omega \to \omega \text{ such as } f \leq \alpha \circ g \text{ and } g \leq \alpha \circ f$$

Cost function : element of $(\omega + 1)^{\mathbb{A}^*}/\approx$.
Extension of the notion of language.

Example

For $\mathbb{A} = \{a, b, c\}$,
$\max(|\cdot|_a, |\cdot|_b) \;\approx\; |\cdot|_a + |\cdot|_b$,
but $|\cdot|_a \;\not\approx\; u \mapsto \max\{n/a^n \text{ factor of } u\}$

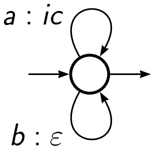
Cost automata

<u>Aim :</u>Calculating cost functions with (nondeterministic) automata.
<u>Idea :</u> Counters which can be modified when a transition is used.
Formally, $\mathcal{A} = \langle Q, \mathbb{A}, In, Fin, \Gamma, \Delta \rangle$ with

$$\Delta \subseteq Q \times A \times (\{i, r, c\}^*)^\Gamma \times Q$$

Actions : increment (i), reset (r), check (c).

Example



Semantics of cost automata

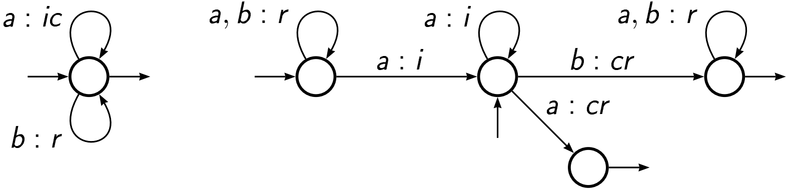
Let $C(e) = \{\text{values of counters checked during the execution } e\}$

$$\begin{aligned} & [\![\mathcal{A}]\!]_B(u) = \inf\{\sup C(e) \ : \ e \text{ run over } u\} \ , \\ \text{and} \quad & [\![\mathcal{A}]\!]_S(u) = \sup\{\inf C(e) \ : \ e \text{ run over } u\} \ . \end{aligned}$$

are two semantics defining respectively B-automata and S-automata.

Examples of cost automata

With one counter, $\mathcal{A}$ (B-automaton) and $\mathcal{A}'$ (S-automaton) :



$[\![\mathcal{A}]\!]_B \approx [\![\mathcal{A}']\!]_S \approx$ block-size with
block-size$(u) = \max\{n \in \omega / a^n$ factor of $u\}$.

Theorem (Colcombet 09)

B-automata and S-automata are equivalent (modulo $\approx$) from the point of view of recognized cost functions.

Temporal automata

Simple automata : Only actions : $\{\varepsilon, ic, r\}$ for B-automata and $\{\varepsilon, i, r, cr\}$ for S-automata.

Temporal automata : Intuitive idea : measuring the time. Only actions : $\{ic, r\}$ for B-automata (Kirsten and Bala's *desert automata*) and $\{i, r, cr\}$ for S-automata.

Example

For $\mathbb{A} = \{a, b\}$, *block-size is temporal, but* $|\cdot|_a$ *is not.*

Theorem

For a cost function, it is equivalent (modulo $\approx$*) to be recognized by*

- *temporal* B*-automaton*
- *temporal* B*-automaton with* 1 *counter*
- *temporal* S*-automaton*
- *temporal* S*-automaton with* 1 *counter*

We say then that the cost function is temporal.

A little proof

Proposition

$|\cdot|_a$ *is not temporal.*

Proof

Let $\mathcal{A} = \langle Q, \mathbb{A}, In, Fin, \{\gamma\}, \Delta\rangle$ temporal B-automaton computing $g \approx_\alpha |\cdot|_a$ for some α.
Let $K > 2|Q| + 1$ and $N > \alpha(K)$.
Let e minimal run of $\mathcal{A}$ over $u = (b^N a)^K$.
We took $K > 2|Q| + 1$, so we can write $u = xvy$, with $|v|_a \geq 2$ and in e, $p \xrightarrow{v} p$.

- Path $a \xrightarrow{i \ldots i} a$ in v, then $g(u) \geq N > \alpha(|u|_a)$, absurd.
- Each path $a \longrightarrow a$ in v contains a reset.
  Then $\forall m, g(xv^m y) \leq 2g(u)$, absurd because $|v|_a > 0$.

Stabilization semigroups

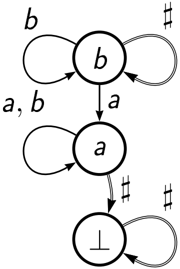
Remind : regular language $\Leftrightarrow$ finite semigroup (Myhill)
and star-free language $\Leftrightarrow$ group-trivial semigroup (Schützenberger)

Stabilization semigroup : $\mathbf{S} = \langle S, \cdot, \leq, \sharp \rangle$, ordered semigroup with a $\sharp$-operator : stabilization over idempotents ($e = e \cdot e$)

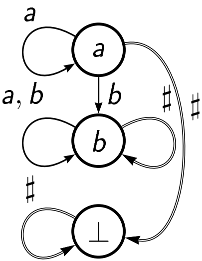
We note $\mathbf{S}^1 = \mathbf{S} \cup \{1\}$ the stabilization monoid associated to $\mathbf{S}$.
Extension of the standard semigroups and monoids.

Semantic through an example

Example : Stabilization semigroups for $|\cdot|_a$ and block-size :



$$\perp \cdot x = x \cdot \perp = \perp$$



Temporal stabilization semigroups

Definition

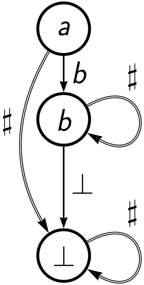
$e \in E(\mathbf{S})$ *is* *stable* *if* $e^\sharp = e$

Definition

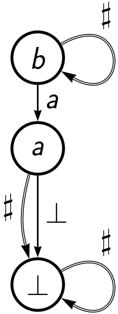
$\mathbf{S}$ *Temporal stabilization semigroup :*

$$\forall e \in E(\mathbf{S}) \text{ stable}, \forall x, y \in \mathbf{S}^1,\ x \cdot e \cdot y \in E(\mathbf{S}) \Rightarrow x \cdot e \cdot y \text{ stable.}$$

block-size :



$|\cdot|_a$:



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A cost function is temporal iff it is recognized by a temporal stabilization semigroup.

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- $\mathbf{S}$ *is computable effectively*

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Let f be a regular cost function,

- *There exists a (quotient-wise) minimal stabilization semigroup* $\mathbf{S}$ *recognizing* f
- $\mathbf{S}$ *is computable effectively*
- f *is temporal iff* $\mathbf{S}$ *is temporal*

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Theorem

Let f be a regular cost function,

- *There exists a (quotient-wise) minimal stabilization semigroup* $\mathbf{S}$ *recognizing f*
- $\mathbf{S}$ *is computable effectively*
- *f is temporal iff* $\mathbf{S}$ *is temporal*

Corollary

It is decidable whether a regular cost function is temporal.

Conclusion

Summary :

- Temporal class defined via cost automata
- Simplifications of constructions in this class
- Characterization by stabilization semigroups
- Minimization of stabilization semigroups
- Decidability of the temporal class