CR 13: Graph Decompositions
Homework 1

Date: 28/09/2021, to hand in by 26/10/2021
Total Marks: 20

• You can handwrite or type your composition.
• The 3 exercises are independent. You may tackle them in any order.
• For each question, you may of course consider the statements of the previous questions as true even if you could not prove them.
• We encourage you to draw figures whenever you feel that they will be useful for your reader.

1 Treewidth of a particular family of planar graphs (3 marks)

Let us consider the planar graph $G_n$ on $3 \cdot 2^n - 2$ vertices obtained from two full binary trees with height $n$, hence $2^n$ leaves, by identifying pairs of homologous leaves and adding a path linking the identified leaves, in a planar way. See figure 1 for an illustration of $G_3$.

![Figure 1: The planar graph $G_3$.](image)

Q.1) Show that for every integer $n \geq 2$, the treewidth of $G_n$ is at least 3 and at most 4. Bonus point, if you show that for every integer $n \geq 4$, the treewidth of $G_n$ is precisely 4. 3+1 marks

2 Maximum $k$-Coverage in planar graphs (9 marks)

Maximum $k$-Coverage generalizes the $k$-Vertex Cover problem by asking whether $k$ vertices touching at least $p$ edges exist. Note that if one sets $p$ to $|E(G)|$, Maximum $k$-Coverage is indeed equivalent to $k$-Vertex Cover. In particular, Maximum $k$-Coverage is NP-complete.

<table>
<thead>
<tr>
<th>Maximum $k$-Coverage</th>
<th>Parameter: $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> A graph $G$ and two positive integers $k$ and $p$.</td>
<td></td>
</tr>
<tr>
<td><strong>Question:</strong> Is there a set $S \subseteq V(G)$ such that $</td>
<td>S</td>
</tr>
</tbody>
</table>
Importantly we consider \( k \) as the parameter, and not \( p \), nor a combination of \( p \) and \( k \). Contrary to \textit{k-Vertex Cover}, \textit{Maximum \( k \)-Coverage} does \textit{not} admit a fixed-parameter tractable (FPT) algorithm in general graphs, i.e., one with running time \( f(k)|V(G)|^{O(1)} \) for some computable function \( f \). The goal of this exercise is to design FPT algorithms for \textit{Maximum \( k \)-Coverage} when restricted to planar graphs.

\textbf{Q.2)} Present an algorithm solving \textit{Maximum \( k \)-Coverage} in time \( 2^{|V(G)|^{O(1)}} \) when the (non-necessarily planar) input \((G,k,p)\) comes with a nice tree decomposition of \( G \) of width \( t \). Detail the correctness only in the case of the introduce node. \hfill 2.5 marks

\textbf{Q.3)} Using the previous question show that \textit{Maximum \( k \)-Coverage} admits a \( 2^{O(k)}|V(G)|^{O(1)} \) time algorithm in planar graphs. \hfill 2 marks

We will now find a faster algorithm with running time \( 2^{O(\sqrt{k})}|V(G)|^{O(1)} \).

\textbf{Q.4)} Explain why the bidimensionality technique (small treewidth or large grid as minor or as edge contraction) does not work \textit{as is} for the \textit{Maximum \( k \)-Coverage} problem. \hfill 0.5 marks

We recall that \( X \subseteq V(G) \) is a dominating set of \( G \) whenever \( N[X] = V(G) \), that is, \( X \) and its neighborhood spans the entire vertex set of \( G \).

\textbf{Q.5)} Given any graph \( G \), find a polytime-computable ordering of its vertices, say, \( v_1, v_2, \ldots, v_{|V(G)|} \), such that if the input \((G,k,p)\) of \textit{Maximum \( k \)-Coverage} has a solution, then it has one solution, \( S \), such that there is an integer \( r \) satisfying both \( S \subseteq \{v_1, \ldots, v_r\} \) and \( S \) is a dominating set of the graph \( G'[\{v_1, \ldots, v_r\}] \), i.e., the subgraph of \( G \) induced by \( \{v_1, \ldots, v_r\} \).

\textit{Hint: the adequate ordering is not proper to planar graphs, and can break ties arbitrarily.} \hfill 2 marks

For the next question, you can use without a proof that, there is a polytime algorithm that, given any planar graph \( G \) and positive integer \( k \), outputs a nice tree decomposition of \( G \) of width \( 20k \) or an edge contraction of \( G \) isomorphic to the \( k \)-by-\( k \) triangulated grid.

\textbf{Q.6)} Deduce an algorithm solving \textit{Maximum \( k \)-Coverage} in planar graphs in \( 2^{O(\sqrt{k})}|V(G)|^{O(1)} \). \hfill 2 marks

\section{3 Breakable permutations (8 marks)}

Let \( \sigma \) be a permutation of the set \( [n] : = \{1, \ldots, n\} \), that is a bijection of \( [n] \) into itself. If \( S \) is a subset of \( [n] \), we denote by \( \sigma(S) \) the set of images of elements of \( S \). Let \( n \leq m \) be two positive integers. A permutation \( \tau \) of \( [n] \) is a \textit{subpermutation} of \( \sigma \) of \( [m] \) if there is an increasing injective function \( f \) from \( [n] \) into \( [m] \) such that for all pairs of distinct \( i, j \) in \([n] \) we have \( \tau(i) < \tau(j) \) if and only if \( \sigma(f(i)) < \sigma(f(j)) \). If such an \( f \) exists then we write \( \tau \prec_s \sigma \).

\textbf{Q.7)} Show that \( \prec_s \) is a partial order. \hfill 0.5 marks

We say that \( \sigma \) of \( [n] \) is \textit{breakable} if \( n = 1 \) or if there exists \( i \in [n - 1] \) such that \( \sigma([i]) = [i] \) or \( \sigma([i]) = [n] \setminus [n - i] \), and if this property also holds for all subpermutations of \( \sigma \).

\textbf{Q.8)} Find all permutations of \( [4] \) which are not breakable. \hfill 1.5 marks

\textbf{Q.9)} Propose an \( O(n^2) \) algorithm which tests if a permutation of \( [n] \) is breakable or not. \hfill 2 marks

\textbf{Q.10)} Show that \( \prec_s \) is a well-quasi-order on the set of breakable permutations. \hfill 4 marks