

The Complexity of Playing Durak

Abstract

Durak is a Russian card game in which players try to get rid of all their cards via a particular attack/defense mechanism. The last player standing with cards loses. We show that, even restricted to the perfect information two-player game, finding optimal moves is a hard problem. More precisely, we prove that, given a generalized durak position, it is PSPACE-complete to decide if a player has a winning strategy. We also show that deciding if an attack can be answered is NP-hard.

1 Introduction

The computational complexity of games is a fruitful research topic which started to formalize in the late seventies [Schaefer, 1978]. From an AI perspective, it offers an insight into what may and may not be computed efficiently in the process of solving a game. The complexity of games has been and is still extensively studied, giving rise to a few tractability results, such as solving in polynomial time NIM [Bouton, 1901] and SHANNON EDGE SWITCHING GAME [Bruno and Weinberg, 1970], and a series of intractability results. For instance, HEX [Reisch, 1981], OTHELLO [Iwata and Kasai, 1994], AMAZONS [Furtak et al., 2005; Hearn and Demaine, 2009], and HAVANNAH [Bonnet et al., 2013a] are PSPACE-complete, while CHESS (without fifty-move rule) [Fraenkel and Lichtenstein, 1981], GO (with Japanese ko rules) [Robson, 1983], and CHECKERS [Robson, 1984], are EXPSPACE-complete.

That list suggests that the computational complexity of board games is relatively well understood. The main motivation of this paper is to go towards a similar understanding for card games. Indeed, although card games are arguably as popular as board games, far less is known concerning their complexity. We only know of a handful of results mostly on trick-taking card games. Bridge (or whist) with two hands and a single suit, or with two hands and mirror\(^1\) suits can be solved in polynomial time [Kahn et al., 1987; Wästlun, 2005a; 2005b]. Some generalizations of bridge with more hands were proven PSPACE-complete [Bonnet et al., 2013b]. Finally, the complexity of problems linked to the games of UNO [Demaine et al., 2010] and SET [Lampis and Mitsou, 2014] has been studied.

Here, we wish to pursue this line of works by investigating the complexity of durak whose game mechanism is not based on taking tricks. Durak is a two to six-player card game intensively played in Russia and East European countries. Durak is the Russian word for fool which designates the loser. There is no winner in durak, there is just a loser: the last player standing with cards. We sketch a simplified version\(^2\) of the rules for two players and without trumps.

The game is played with 36 cards, by keeping the cards from the sixes (lowest cards) to the aces (highest cards) in a standard 52-card deck. Both players, let us call them P and O, are dealt a hand of six cards and their goal is to empty their hand before the opponent does. The remaining cards form the pile. The game is made of rounds. A designated player, say P, leads the first round by playing any card c of his hand. In this round, P is the attacker, O is the defender, and c is the first attacking card. The defender can skip, at any time. In that case, the defender picks up all the cards played during the round (by both players) and puts them into his hand; then, the attacker remains the attacker for the next round. The defender can also defend the current attacking card by playing a higher card in the same suit. Each time his opponent defends, the attacker can (but is not forced to) play an additional attacking card (up to a limit of six cards) provided it has the same rank as a card already played during the round (by either himself or his opponent). If the defender does defend all the attacking cards played by the attacker, all the cards played during the round are discarded and the defender leads the next round, thereby becoming the new attacker. After each round, any player with less than six cards, draws cards in the pile until he reaches the total of six.

In fact, we will consider that the pile is empty and that the two players have perfect information. Why do we make those assumptions? In durak, one does not win but has to avoid losing. While the pile is not empty, or while there are three players or more still in the game, the risk of quickly losing is relatively weak. This is one motivation for focusing

\(^1\)A suit is said mirror whenever both players have the same number of cards in it.

\(^2\)For a full description of the rules of Durak, see http://www.pagat.com/beating/podkidnoy_durak.html
on the two-player game with an empty pile. Now, from his hand and the cards played and discarded so far, a player can infer the hand of his opponent, yielding perfect information. More importantly, we almost exclusively prove negative results, and our hardness proofs do not require more than two players, nor a non empty pile, nor trumps.

After precisely the notations, the vocabulary and the rules of durak in Section 2, we show that deciding if one player can defend any attack is NP-hard, in Section 3. The main result of the paper is the PSPACE-hardness of two-player perfect information durak and is presented in Section 4. Our reduction (from 3-TQBF) requires the introduction of several notions: weaknesses, well-covered weaknesses, and strong suits. We believe that those notions can be of importance in designing good artificial players for durak.

2 Preliminaries

For any integers \( x < y, \{ x, y \} := \{ x, x+1, \ldots, y-1, y \} \) and \( [x] := \{ 1, x \} \). A card is defined by a suit symbol \( s_j \) and an integer \( i \) called rank, and is denoted by \( (s_j, i) \). A hand is a set of cards.

Example 1. \( h_1 = \{(s_2, 1), (s_3, 1), (s_3, 5), (s_5, 1), (s_5, 3), (s_5, 4), (s_6, 5)\} \) is a hand. Card \( (s_2, 1) \) has rank 1 in suit \( s_2 \).

Definition 1. A durak position \( \mathcal{P} = (h(P), h(O), L, y) \) is given by two hands \( h(P) \) and \( h(O) \) of \( P \) and \( O \), an indicator \( L \in \{ P, O \} \) of who leads the next round (equivalently, whose turn it is) and a threshold \( y \), that is the maximum number of attacking cards allowed in a round.

Rules. Relation \( \preceq \) defines a partial order over the cards by: for any suit \( s_j \) and any \( i_1, i_2 \in [r] \), \( (s_j, i_1) \preceq (s_j, i_2) \) if \( i_1 \leq i_2 \). If \( c_1 \preceq c_2 \) and \( c_1 \neq c_2 \), we write \( c_1 \prec c_2 \).

A game from an initial position \( \mathcal{P} = (h(P), h(O), L, y) \) is composed of rounds that are themselves composed of moves. If \( h(P) = \emptyset \) or \( h(O) = \emptyset \) the game ends, the player still having cards loses, and his opponent wins. We assume that \( P \) is the current attacking player (i.e., \( L = P \)). If \( c_1, c_2, \ldots, c_p \) is the list of attacking cards played by \( P \), so far, and \( d_1, d_2, \ldots, d_{p-1} \) the list of defending cards played by \( O \) then \( p \leq y \), and for each \( i \in [p-1], c_i \prec d_i \) and \( c_{i+1} \) has the same rank as at least one card in \( [c_1, d_1, \ldots, c_i, d_i] \).

\( O \) can skip. In that case, we say that \( O \) takes the cards. \( P \) can add extra attacking cards \( c_{p+1}, \ldots, c_q \) (with \( p + 1 \leq q \leq y \)) provided that they are of the same rank as a card in \( [c_1, c_2, \ldots, c_{p-1}, d_{p-1}, d_p] \). The next position is \( (h(P) \setminus \{ c_1, \ldots, c_q \}, h(O) \cup \{ c_1, \ldots, c_q \}, P, y) \).

\( O \) can also try to defend by playing a card \( d_p \) such that \( c_p \prec d_p \). In that case, \( P \) can continue the attack (if \( p < y \)) or skips. If the attacker skips, the next position is \( (h(P) \setminus \{ c_1, \ldots, c_p \}, h(O) \setminus \{ d_1, \ldots, d_p \}, O, y) \). The cards played during the round are discarded, \( O \) has defended until the end, and \( O \) takes the lead. When a player plays a series of attacking cards that cannot be defended by the opponent, we say that he gives those cards to his opponent.

A draw occurs if \( h(P) = h(O) = \emptyset \).

3 On Defending an Attack

Defending until the end if possible, and taking the first attacking card otherwise, constitutes a decent heuristic for the defender. Unfortunately, we show that deciding if a defense is possible is already a hard problem.

Theorem 1. Given a durak position \( \mathcal{P} \), deciding if \( P \) can defend any attack of \( O \) until the end is \( NP \)-hard.

Proof. We reduce from \( 3\text{-SAT} \). Let \( \mathcal{C} = \{C_1, \ldots, C_m\} \) be any instance of \( 3\text{-SAT} \), where each \( C_i \) is a 3-clause over the set of variables \( X = \{ x_1, \ldots, x_n \} \). We construct a durak position \( \mathcal{P} = (h(P), h(O), O, \infty) \) with \( n + m \) suits, \( 2n + 3 \) ranks, and...
3n + 5m cards in total (O has n + 3m cards and P has 2n + 2m cards) such that \( C \) is satisfiable iff \( P \) can defend until the end any attack of \( O \). Let \( r : \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\} \rightarrow [2, 2n + 1] \) such that \( r(x_i) = 2i \) and \( r(\neg x_i) = 2i + 1 \) for all \( i \in [n] \), and \( l : [2, 2n + 1] \rightarrow \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\} \) be the inverse function.

For each variable \( x_i \ (i \in [n]) \), we devote a suit \( s_{x_i} \) where \( O \) has the card \((s_{x_i}, 1)\) and \( P \) has the two cards \((s_{x_i}, 2i)\) and \((s_{x_i}, 2i + 1)\). For each clause \( C_j = l_1 \lor l_2 \lor l_3 \), we devote a suit \( s_{C_j} \) where \( O \) has the three cards \((s_{C_j}, r(l_1)), (s_{C_j}, r(l_2)), (s_{C_j}, r(l_3))\), and \((s_{C_j}, r(l))\), while \( P \) has the two cards \((s_{C_j}, 2n + 2)\) and \((s_{C_j}, 2n + 3)\). This ends the construction (see Figure 2).

First, we may observe that if \( O \) starts the attack with a card \((s_{C_j}, u)\), the defense is easy since \( P \) can follow this family of variations: \((s_{C_j}, u)(s_{C_j}, 2n + 2), (s_{C_j}, u)(s_{C_j}, 2n + 2), \ldots, (s_{C_j}, u)(s_{C_j}, 2n + 2), (s_{C_j}, u)(s_{C_j}, 2n + 2)\), where each \( k_i \ (i \in [h]) \) is the index of a clause where literal \( l(u) \) appears. The only remaining attempt for \( O \) is to start attacking with a card \((s_{x_i}, 1)\), for some \( i \in [n] \).

If \( \forall \) is satisfiable, we fix a satisfying assignment \( a : X \rightarrow \{\top, \bot\} \). Symbol \( \top \) (respectively \( \bot \)) is interpreted as setting the variable to \( true \) (respectively \( false \)). \( P \) can defend the attack in the following way. On each attacking card \((s_{x_i}, 1)\) \((i \in [n])\), \( P \) plays \((s_{x_i}, 2i + 1)\) if \( a(x_i) = \top \) and plays \((s_{x_i}, 2i)\) if \( a(x_i) = \bot \). Now, in each suit \( s_{C_j} \), \( O \) can attack with at most two cards, and \( P \) can defend with \((s_{C_j}, 2n + 2)\) and \((s_{C_j}, 2n + 3)\). Indeed, if there is a suit \( s_{C_j} \) where \( O \) can play his three cards of rank, say, \( u_1, u_2, \) and \( u_3 \), then no literal among \( l(u_1), l(u_2) \), and \( l(u_3) \) would be set to true by assignment \( a \), so the clause \( C_j \) would not be satisfied.

If \( \forall \) is not satisfiable, no assignment \( a : X \rightarrow \{\top, \bot\} \) satisfies every clauses. In particular, after \( O \) attacks with all the cards \((s_{x_i}, 1)\) \((i \in [n])\) and \( P \) has to defend with \((s_{x_i}, u)\) \((u_i \in [2i, 2i + 1])\), the assignment defined by \( a(x_i) = \top \) if \( u_i = 2i + 1 \) and \( a(x_i) = \bot \) if \( u_i = 2i \), does not satisfy some clause \( C_j \). Thus, \( P \) has played cards of rank \( r(l_1), r(l_2) \), and \( r(l_3) \) where \( C_j = l_1 \lor l_2 \lor l_3 \). Hence, \( O \) can attack with the three cards \((s_{C_j}, r(l_1)), (s_{C_j}, r(l_2)), \) and \((s_{C_j}, r(l_3))\), and \( P \) can not defend, since he has only two cards in the suit \( s_{C_j} \). 

4 On Playing Optimally

**Proposition 1.** Given a durak position \( \mathcal{D} \), deciding if \( P \) has a winning strategy is in \( \mathcal{P} \mathcal{S} \mathcal{A}\)PCE.

**Proof.** We have to show that the length of a game is polynomially bounded by the size of the instance, or equivalently by the total number \( n \) of cards in \( \mathcal{D} \). Then, we can conclude by doing a depth-first minimax search. A player cannot have the lead on \( n \) consecutive rounds. Indeed, when a player keeps the lead, at least one card is transferred, at each round, from his hand to the hand of his opponent. So, if a player keeps the lead for \( n - 1 \) consecutive rounds, he wins. When the lead goes from a player to his opponent, at least two cards are discarded. Thus, a game cannot contain more than \( \frac{n(n - 1)}{2} \) rounds. A round lasts at most \( n + 1 \) moves, so the game length is bounded by \( (n - 1)n(n + 1)/2 = O(n^3) \).

We now need some extra definitions and observations.

**Definition 2.** A weakness for player \( P \) is a rank \( i \in [r] \) satisfying the following two conditions: (1) \( h(P) \) contains at least one card of rank \( i \), and (2) for each suit \( s_j \) with \((s_j, i) \in h(P)\), there is a rank \( i' > i \) such that \((s_j, i') \in h(O)\).

Informally, \( P \) has each of his cards of rank \( i \) dominated by a card of \( O \). The set of cards of rank \( i \) in \( h(P) \) is also called weakness and each card of the set is called weakness card. A rank \( i \) which is not a weakness for \( P \), or the set of cards of rank \( i \) in \( h(P) \) is called a non weakness (for \( P \)).

Assuming that the threshold \( y \) is greater than the total number of cards of rank \( i \), for any \( i \in [r] \), (we will refer to this assumption as \( \mathcal{A} \mathcal{J} \) in what follows) we may observe that player \( P \), at his turn, can give all his cards of rank \( i \) to \( O \), provided that \( i \) is a non weakness for \( P \). Indeed, by definition, there is a suit \( s_j \) such that \((s_j, i) \in h(P)\) and no card \( c \in h(O) \) satisfies \((s_j, i) < c \). Thus, \( O \) cannot defend this attack. Therefore, we can show the following.
Lemma 1. Under (\mathcal{AC}), if \(P\), at his turn, has only one weakness \(i\), then he has a winning strategy.

Proof. If \(i'\) is a non weakness for \(P\), and \(P\) gives to \(O\) all his cards of rank \(i < i'\), then \(i'\) is still a non weakness for \(P\) in the resulting position. So, \(P\) wins by giving all his non weaknesses to \(O\) by increasing ranks and finally plays all his cards of rank \(i\).

Definition 3. A strong suit for player \(P\) is a suit \(s_j\) where he has at least one card and \(O\) has none.

We observe that the rank of any card in a strong suit of \(P\) is a non weakness for \(P\). We say that \(P\) can win by attacking only if he has a winning strategy such that \(O\) can never take the lead.

Example 3. Let \(\mathscr{P} = \langle h(P) = \{(s_1, 1), (s_2, 2)\}, h(O) = \{(s_1, 2), (s_2, 1), (s_3, 3)\}, \infty \rangle\). \(P\) can win by attacking only due to the variations: (a) \((s_1, 1)(s_2, 1)\); (b) \((s_1, 1)(s_1, 2)\); (c) \((s_1, 1)(s_2, 2)(s_3, 3)\); and (d) \((s_1, 1)(s_2, 2)\). Note that if \(O\) had the lead in \(\mathscr{P}\), then he would win by Lemma 1 since he only has 1 as a weakness.

The following lemma is very useful to reduce the number of potentially good first attacking card. Intuitively, it says that if you cannot win by attacking only, it is useless (and possibly harmful) to give cards to your opponent that he will be able to give you back when he will have the lead.

Lemma 2. Under (\mathcal{AC}), if \(P\) has a winning strategy but cannot win by attacking only, \(P\) has a card \((s_j, i)\) in a strong suit \(s_j\), and \(i\) is a non weakness for \(P\), then \(P\) has a winning strategy that does not start the round with cards of rank \(i\).

Proof. \(O\) can accept to take the set \(S\) of cards of rank \(i\) played by \(P\). This cannot make that \(P\) is now winning by attacking only since \(i\) is a non weakess for \(P\). So, \(P\) could have forced \(O\) to take all his cards of rank \(i\) at any moment. Thus, \(O\) will eventually get the lead back. By definition, \(P\) has no card in the strong suit \(s_j\) of \(O\). It implies that \(O\) has not been attacked in \(s_j\), so he has exactly the same cards in \(s_j\) as in the initial position. In particular, \((s_j, i)\) \(\in h(O)\) and \(O\) can give \(S\) back to \(P\), making the first attack of \(P\) useless.

There is quite a lot of conditions in Lemma 2, and checking that \(P\) cannot win by attacking only, to know if the lemma applies, may be problematic. Therefore, we give a sufficient condition implying that a player cannot win by attacking only.

Definition 4. A well-covered weakness for \(P\) is a weakness \(i\) such that for each \((s_j, i)\) \(\in h(P)\), there is a higher card \((s_j, i')\) \(\in h(O)\) and \(P\) has no card of rank \(i'\).

Intuitively, if \(P\) attacks with a well-covered weakness, \(O\) can defend so that \(P\) cannot play any other attacking card at this round.

Lemma 3. If \(P\) has two well-covered weaknesses, he cannot win by attacking only.

Proof. Let \(i_1 \neq i_2\) be the two well-covered weaknesses for \(P\). First, we remark that while \(P\) gives cards to \(O\) which are not of rank \(i_1\) or \(i_2\), they remain well-covered weaknesses. So, \(O\) takes any cards of rank \(i \notin \{i_1, i_2\}\) without trying to defend. At some point, \(P\) has to start an attack with cards of rank \(i_1\) or \(i_2\). In both cases, \(O\) can defend until the end, by definition of a well-covered weakness.

The proof of the following lemma is similar to the proof of Lemma 2 and therefore omitted.

Lemma 4. Under (\mathcal{AC}), if \(P\) has a winning strategy but cannot win by attacking only, then \(P\) has a winning strategy that does not start the round with the highest card of some suit.

Theorem 2. Given a durak position \(\mathscr{P}\), deciding if \(P\) has a winning strategy is \(\text{PSPACE}\)-complete.

Proof. It is in \(\text{PSPACE}\) by Proposition 1. We show that it is \(\text{PSPACE}\)-hard by a reduction from the \(\text{PSPACE}\)-hard problem \(\text{QBF}\) which remains so even if all the variables are quantified, the quantifiers alternate starting with \(\exists\) and ending with \(\forall\). This restricted problem is sometimes called 3-\(\text{QBF}\) and consists of deciding whether \(\exists x_1 \forall x_2 \exists x_3 \ldots \forall x_m \phi(x_1, x_2, \ldots, x_n)\) is true or false, where \(\phi\) is a conjunction of clauses with three literals. We fix a 3-CNF formula \(\phi\) with \(m\) clauses \(C_1, C_2, \ldots, C_m\). We will build a durak position \(\mathscr{P} = \langle h(P), h(O), P.2(n + m + 1) \rangle\) with \(3n + 16\) ranks, \(6m + \frac{11}{3}n + 8\) suits, and \(26m + \frac{32}{3}n + 24\) cards such that \(\psi = \exists x_1 \forall x_2 \exists x_3 \ldots \forall x_m \phi\) is true iff \(P\) has a winning strategy from the position \(\mathscr{P}\). For technical reasons that will become relevant later, we define \(\psi' = \exists x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \forall x_m \phi'\), where \(\phi'\) is the conjunction of the \(2m\) clauses \(x_0 \lor C_1 \lor x_2 \lor C_2, \ldots, x_0 \lor C_{m-1} \lor x_0 \lor C_m, \lor x_0 \lor C_{m-1} \lor \neg x_0 \lor C_m\). We denote \(x_0 \lor C_i \lor \neg x_0 \lor C_{i'} \lor \neg x_0 \lor C_i' \lor C_i''\) for all \(i \in [m]\). We observe that \(\psi'\) is true iff \(\psi\) is true, and \(\phi'\) is a conjunction of 4-clauses.

Existential quantifier gadget. For each odd \(i \in [n]\), we encode \(\exists x_i\) by devoting four suits \(s^+_{1i}, s^+_{2i}, s^+_{3i}, s^+_{4i}\) where \(P\) has four cards: \((s^+_{1i}, o_i)\), \((s^+_{2i}, o_i)\), \((s^+_{3i}, o_i + 1)\), and \((s^+_{4i}, o_i + 2)\) and \(O\) has four cards: \((s^-_{1i}, o_i + 1)\), \((s^-_{2i}, o_i + 2)\), \((s^-_{3i}, o_i + 1)\), and \((s^-_{4i}, o_i + 3)\). We set \(o_i := 3i + 7\). Figure 4a displays the geometric representation of the existential gadget and the two local outcomes if \(P\) decides to set \(x_i\) to true (Figure 4b) or to set \(x_i\) to false (Figure 4c).

Universal quantifier gadget. For each even \(i \in [n] \cup \{0\}\), we encode \(\forall x_i\) by devoting three suits \(s^0_{1i}, s^0_{2i}, s^0_{3i}\) where \(P\) has three cards: \((s^0_{1i}, o_i)\), \((s^0_{2i}, o_i + 1)\), and \((s^0_{3i}, o_i + 2)\) and \(O\) has four cards: \((s^0_{1i}, o_i + 1)\), \((s^0_{2i}, o_i + 2)\), \((s^0_{3i}, o_i + 3)\), and \((s^0_{4i}, o_i + 3)\).
\[ \phi(x_1, x_2, x_3) = (\forall x_1) \land (\forall x_2) \land (\forall x_3) \land (\forall x_4) \land \ldots \]

We define the rank \( r(l) \) of literal \( l \) as \( 3i + 8 \) if \( l = x_i \) or \( 3i + 9 \) if \( l = \neg x_i \). We denote by \( s(l) \) the suit wherein \( O \) has a card of rank \( r(l) \) in the gadget associated to the quantified variable \( x_i \) with \( l \in \{ x_i, \neg x_i \} \). So, if \( x_i \) is universally quantified, then \( s(x_i) = s(\neg x_i) = s_1 \) while if \( x_i \) is existentially quantified, then \( s(x_i) = s_1 \) and \( s(\neg x_i) = s_2 \).

For each 4-clause \( C = l_1 \lor l_2 \lor l_3 \lor l_4 \) of \( \phi \), we denote a suit \( s_C \) for \( \forall \) and \( s_{\neg C} \) for \( \exists \). Player \( P \) has the 4 cards \( (s_C, l_1(l_1)), (s_C, l_2(l_2)), (s_{\neg C}, l_3(l_3)), \) and \( (s_{\neg C}, r(l_4)) \) while \( O \) has the 5 cards \( (s_C, 5) \) and \( (s_C, k) \) for \( k \in [3n + 13, 3n + 16] \).

\( P \) has to start the attack with \( (s_1^0, 7) \). We first show that \( P \) has to start the attack with \( (s_1^0, 7) \) with the idea of getting rid of the two weaknesses 7 and 3n + 10 in the same round.

In case of the second or third option, \( O \) can defend until the end: \( (s_{d_4}^k, 3)(s_{d_4}^k, 4), \) or \( (s_{d_4}^k, 3n + 10)(s_{d_4}^k, 3n + 11), \) \( \ldots (s_{d_4}^k, 3n + 10)(s_{d_4}^k, 3n + 11), \) \( \ldots \) and \( O \) wins with the following strategy, which we denote by \( \mathcal{P} \). Player \( P \) starts the next round with all his cards \( (s_C, 5) \) for each clause \( C \) of \( \phi \). \( P \) has to defend, since otherwise \( O \) leads the next round with a single weakness, so \( O \) wins by Lemma 1. In particular, \( P \) should defend the card \( (s_{C_1}^*, 5) \). The only way to do so is to play \( (s_C, r(l)) \) where \( l \) appears in \( C_1 \). If \( l = x_0 \) the winning continuation for \( O \) is \( (s_{C_1}^*, 5)(s_{C_1}^*, r(x_0)), (s_0, r(x_0))(s_{C_1}^*, 5)(s_0, r(x_0)), \ldots \), whereas, if \( l \neq x_0 \) the variation is \( (s_{C_1}^*, 5)(s_{C_1}^*, r(l)), (s_0, r(l))(s_{C_1}^*, 5)(s_0, r(l)), \ldots \), and in both cases \( O \) leads the next round with only one weakness.

Finally, starting an attack with a card of rank 1 cannot help \( P \); \( O \) would just skip. Indeed, let \( S \) be the set of cards of rank 1 played by \( P \) and taken by \( O \). Either \( S \neq \{(s_0, 1)\} \), and \( O \) can give all those cards back to \( P \) the next time he takes the lead. Either \( S = \{(s_0, 1)\} \), but \( P \) could give this card to \( O \) any time he is the attacker.

\( P \) cannot play cards of \( s_p \). We show that during the first round starting with \( (s_0^0, 7) \), \( P \) loses if he plays a card \( (s_p, l) \). Assume \( P \) does. \( O \) has to take all the cards played during the round, in particular \( (s_0, 7) \). It is a new weakness for \( O \), but \( P \) has also a new weakness because he played \( (s_p, l) \): 1. So, each player has 3 weaknesses and \( P \) has still the lead. However, \( O \) wins in the following way. While \( P \) starts attacks...
with non weaknesses, O skips. Again, we observe that, as the position is, this step cannot create weaknesses for O. When P starts an attack with a weakness, O defends until the end. This is possible since 3 and 3n+10 are well-covered weaknesses, and P has 2(m+n+1) cards of rank 1, so he would be allowed to add at most 1 extra attacking card, due to the threshold 2(m+n+1). One can check that O can always defend this card of rank i.

Thus, O will take the lead at least twice. The first time O takes the lead, he attacks with (s_0^1, 7). P has to defend, otherwise O wins thanks to strategy \( \mathcal{F} \). The second time, O is left with weaknesses 5 and 1, and wins with \( \mathcal{F} \).

**O should defend until the end.** We show that if P does not play a card of the suit \( s_p \) during this first round, then O should defend until the end. Suppose O skips at some point. Player O takes in his hand the cards played during the round; in particular, the card \((s_0^0, 7)\) which is now a weakness card for O, since P has the card \((s_0^0, 3n+12)\) that cannot have been played in the previous attack of P, for it is the only card with rank 3n+12. P can win by playing \((s_0^3, 3)\) in the next round. O has to defend, since otherwise P is left with only one weakness 3n+10 and wins by Lemma 1. So, the continuation is \((s_0^3, 3)(s_0^2, 4)\). Now, O leads the round and has 3 weaknesses: 1, 5 and 7. Cards \((s_0^1, 7)\) and \((s_0^1, 1)\) are well-covered weakness cards for O. P can skip on all the attacks of O until one of these cards is played. Then, he defends and wins by Lemma 1, since O cannot give cards to P that would constitute weaknesses for P.

**P and O simulates Q8.** If P does not play all his cards of rank 3n+10 during the first round, and O defends until the end, then O wins. O starts the next round with \((s_0^3, 1)\). P has to defend: \((s_0^3, 1)(s_0^3, 6)\); otherwise O wins by Lemma 1. Then, P has the lead, but O wins since he has only one weakness (5), P has two well-covered weaknesses (3 and 3n+10), and P cannot give cards to O which would be new weaknesses for O.

Besides, as P cannot play cards of the suit \( s_p \), one can check that O will be able to defend until the end (thanks notably to cards \((s_c, k) \in h(O)\) for \( k \in \{3n+13, 3n+16\}\). So, P has to find a way of playing all his cards of rank 3n+10. Therefore, due to the threshold \(2(m+n+1)\), P can only play one card of rank \( r(x_i) - 1 \) in each existential gadget \( \exists x_i \). Thus, the first round should be of this form:

\[
\begin{align*}
(s_0^0, 7)(s_0^1, r(\sigma(x_0)))](s_0^0, r(\sigma(x_0)))](s_0^0, r(\neg x_0) + 1), (s_0^1, r(\neg x_0) + 1)(s_0^1, r(\sigma(x_0))), (s_0^1, r(\neg x_0) + 1)(s_0^1, r(\sigma(x_0))), (s_0^1, r(\neg x_0) + 1)(s_0^1, r(\sigma(x_0))), (s_0^1, r(\neg x_0) + 1)(s_0^1, r(\sigma(x_0))),
\end{align*}
\]

for each even k (resp. odd k), \( \sigma(x_1) \in \{x_1, \neg x_1\} \) corresponds to the literal that is set to true by O (resp. P), and \( l_k \in \{1, 2\} \) is the matching index. As in Figure 4 and 5, we interpret the card c of rank in \( \{r(x_i), r(\neg x_i)\} \) played by O (and discarded at the end of the round) as setting \( x_i \) to true if the rank of c is \( r(x_i) \) and as setting \( x_i \) to false if the rank of c is \( r(\neg x_i) \).

Player O leads the next round. At this point, O has still his two weaknesses: 1 and 5; while P has only one weakness: 3.

**If \( \psi \) is false, O wins.** We recall that \( \psi \) and \( \psi' \) are equivalent. Let us assume \( \psi' \) is false. Then, O had a strategy in the first round ensuring that there is a clause \( C'_i = x_0 \lor l'_1 \lor l'_2 \lor l'_3 \) such that \( (s_1^1, l_1), (s_2^1, l_2), (s_3^1, l_3) \) are still in \( h(O) \). O plays all his cards of rank 5. By Lemma 1, P has to defend. In particular, he has to defend on the card \((s_c^1, 5)\). To do so, P can either play \((s_c^1, r(x_0)) \) or \((s_c^1, r(\neg x_0)) \) for some \( k \in \{1, 2, 3\} \). In the former case, the continuation is \((s_c^1, 5)(s_c^1, r(x_0)), (s_c^1, r(\neg x_0)) \) and O adds as extra attacking cards all his cards of rank 5 and potentially his card \((s_0^1, r(\neg x_0)) \). In the latter case, the continuation is \((s_c^1, 5)(s_c^1, r(\neg x_0)), (s_c^1, r(x_0)) \) and again, O gives all his cards of rank 5 to P. In both cases, O wins by Lemma 1.

**If \( \psi \) is true, P wins.** No matter which cards O gives to P, P will not have additional weaknesses. Thus, if P can defend an attack of O until the end, P wins by Lemma 1 (provided that O has still at least one card left). This is equivalent to saying that if O has a winning strategy, he wins by attacking only. Let us show that O cannot win by attacking only.

The last attack of O should be \((s_0^0, 5)(s_0^3, 2)\); while all his other cards have been previously given to P. At some point, O will have to play his weakness cards of rank 5. If O has already given \((s_0, r(x_0)) \) and \((s_0, r(\neg x_0)) \) to P, prior to this attack, then P can defend: 

\[
\begin{align*}
(s_c^1, 5)(s_c^1, r(x_0)), (s_c^1, 5), (s_c^1, r(\neg x_0)), (s_c^1, 5)(s_c^1, r(x_0)), (s_c^1, 5)(s_c^1, r(\neg x_0)),
\end{align*}
\]

this is why we introduced the dummy variable \( x_0 \) and P wins.

So, we can assume that \((s_0, r_0) \) is still in \( h(O) \) when O starts the attack with cards of rank 5, with \( r_0 \in \{r(x_0), r(\neg x_0)\} \). As \( \psi \) is true, P had a strategy in the first round such that, for each clause \( C'_i = x_0 \lor l'_1 \lor l'_2 \lor l'_3 \) \( \forall i \in \{m\} \), there exists \( k_i \in \{1, 2, 3\} \) satisfying \((s_1^0, l_1), (l_1^0, k_i) \) in \( h(O) \). Thus, P can defend like this:

\[
\begin{align*}
(s_c^1, 5)(s_c^1, r(1^0, k_i)), (s_c^1, 5)(s_c^1, r(1^0, k_i)), (s_c^1, 5)(s_c^1, r(1^0, k_i)), (s_c^1, 5)(s_c^1, r(1^0, k_i)),
\end{align*}
\]

and O, to continue the attack, has to play a card \((s_0, r(l_1^0)) \) for some i \( \in \{m\} \). P takes all the cards played during this round. Now, O has a new well-covered weakness \( r_0 \) since \((s_0, r_0) < (s_0, r(l_1^0)) \), \((s_0^0, 5)(s_0^3, 3n+12) \) and O has no card of rank \( r(l_1^0) \) nor \( 3n+12 \). O has two well-covered weaknesses 1 and \( r_0 \). So, by Lemma 3, O cannot win by attacking only, and by the previous remarks, O loses.

**5 Perspectives**

Our proof of \( \text{PSPACE} \)-hardness for two-player durak relies on a finite threshold. One could look for a reduction which does not use the threshold feature. In our opinion, more interesting now would be to establish some polynomial fragments of the game. For instance, we ask as an open question if the seemingly very simple two-player durak with a single suit is solvable in polynomial time.
References


