Twin-width and ordered binary structures

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Trigraphs

Three outcomes between a pair of vertices: edge, or non-edge, or red edge (error edge)
Contractions in trigraphs

Identification of two non-necessarily adjacent vertices
Contractions in trigraphs

Identification of two non-necessarily adjacent vertices
Contractions in trigraphs

edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing
Contraction sequence

A contraction sequence of $G$:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
$G_i$ is obtained by performing one contraction in $G_{i+1}$. 
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Sequence of trigraphs \( G = G_n, G_{n-1}, \ldots, G_2, G_1 \) such that
\( G_i \) is obtained by performing one contraction in \( G_{i+1} \).
Twin-width

tww(G): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have maximum red degree at most $d$.

Maximum red degree $= 0$

overall maximum red degree $= 0$
tww(G): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have maximum red degree at most $d$.

Maximum red degree = 2
overall maximum red degree = 2
Twin-width

tww$(G)$: Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have \textit{maximum red degree} at most $d$.

Maximum red degree $= 2$
 overall maximum red degree $= 2$
Twin-width

tww(\(G\)): Least integer \(d\) such that \(G\) admits a contraction sequence where all trigraphs have \textit{maximum red degree} at most \(d\).

Maximum red degree = 2
overall maximum red degree = 2
Twin-width

tww($G$): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have maximum red degree at most $d$.

![Diagram]

Maximum red degree = 1
overall maximum red degree = 2
Twin-width

tww($G$): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have \textit{maximum red degree} at most $d$.

Maximum red degree = 1

\textit{overall maximum red degree} = 2
Twin-width

tww(\(G\)): Least integer \(d\) such that \(G\) admits a contraction sequence where all trigraphs have \textit{maximum red degree} at most \(d\).

Maximum red degree = 0
overall maximum red degree = 2
Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one apex: at most “doubles”
- substitution $G(v \leftarrow H)$: max of the twin-width of $G$ and $H$
Theorem (B., Geniet, Kim, Thomassé, Watrigant ’20 & ’21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time:

- Bounded rank-width, and even, boolean-width graphs,
- every hereditary proper subclass of permutation graphs,
- posets of bounded antichain size,
- unit interval graphs,
- $K_t$-minor free graphs,
- map graphs with embedding,
- $d$-dimensional grids,
- $K_t$-free unit $d$-dimensional ball graphs,
- $\Omega(\log n)$-subdivisions of all the $n$-vertex graphs,
- cubic expanders defined by iterative random 2-lifts from $K_4$,
- flat classes,
- subgraphs of every $K_{t,t}$-free class above,
- first-order transductions of all the above.
First-order model checking on graphs

**Graph FO Model Checking**

**Parameter:** $|\varphi|$

**Input:** A graph $G$ and a first-order sentence $\varphi \in FO(\{E\})$

**Question:** $G \models \varphi$?
First-order model checking on graphs

**Graph FO Model Checking**

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**Question:** $G \models \varphi$?

Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \forall y \ (E(x, y) \Rightarrow \bigvee_{1 \leq i \leq k} x = x_i \lor y = x_i)$$

$G \models \varphi \iff k$-Vertex Cover
First-order model checking on graphs

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**Question:** $G \models \varphi$?

Example:

$$\varphi = \exists x_1 \exists y_1 \cdots \exists x_k \exists y_k \quad \bigwedge_{\{x,y\} \in \{x_1,y_1,\ldots,x_k,y_k\}} \ x \neq y$$

$$\quad \bigwedge E(x, y) \iff \bigvee_{1 \leq i \leq k} (x = x_i \land y = y_i) \lor (x = y_i \land y = x_i)$$

$G \models \varphi$? $\iff$
First-order model checking on graphs

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$$\land E(x,y) \iff \bigvee_{1 \leq i \leq k} (x = x_i \land y = y_i) \lor (x = y_i \land y = x_i)$$

$G \models \varphi \iff \text{k-Induced Matching}$
First-order model checking on graphs

**Graph FO Model Checking**

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Example:

$$\varphi = \bigvee_{1 \leq q \leq k, \ q \text{ is odd}} \exists x_1 \notin \{s\} \ E(s, x_1) \land (\forall x_2 \notin \{s, x_1\} \neg E(x_1, x_2) \lor$$

$$(\exists x_3 \notin \{s, x_1, x_2\} \ E(x_2, x_3) \land (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

$$\land (\forall x_{q+1} \neg E(x_q, x_{q+1}) \lor x_{q+1} \in \{s, x_1, \ldots, x_q\}))) \cdots )))$$

$G \models \varphi \iff$
First-order model checking on graphs

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$G \models \varphi? \iff \text{Short Generalized Geography}$
FO interpretations and transductions

**FO simple interpretation:** redefine the edges by a first-order formula

φ(x, y) = ¬E(x, y)  
φ(x, y) = E(x, y) ∨ ∃zE(x, z) ∧ E(z, y)  

Theorem (B., Kim, Thomassé, Watrigant ’20)

Transductions of bounded twin-width classes have bounded twin-width.
FO interpretations and transductions

**FO simple interpretation:** redefine the edges by a first-order formula
\[ \varphi(x, y) = \neg E(x, y) \] (complement)
\[ \varphi(x, y) = E(x, y) \lor \exists z E(x, z) \land E(z, y) \] (square)

**FO transduction:** color by \( O(1) \) unary relations, interpret, delete

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![Diagram of FO transduction]
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**FO transduction:** color by \( O(1) \) unary relations, interpret, delete

\[ \varphi(x, y) = E(x, y) \lor \left( G(x) \land B(y) \land \neg \exists z R(z) \land E(y, z) \right) \]
\[ \lor \left( R(x) \land B(y) \land \exists z R(z) \land E(y, z) \land \neg \exists z B(z) \land E(y, z) \right) \]
FO interpretations and transductions

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**FO transduction:** color by \( O(1) \) unary relations, interpret, delete

\[ \varphi(x, y) = E(x, y) \lor (G(x) \land B(y) \land \neg \exists z R(z) \land E(y, z)) \]
\[ \lor (R(x) \land B(y) \land \exists z R(z) \land E(y, z) \land \neg \exists z B(z) \land E(y, z)) \]
FO interpretations and transductions

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**FO transduction:** color by $O(1)$ unary relations, interpret, delete

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Theorem (B., Kim, Thomassé, Watrigant '20)

*Transductions of bounded twin-width classes have bounded twin-width.*
Dependence and monadic dependence

A class $\mathcal{C}$ is **dependent**, if the hereditary closure of every simple interpretation of $\mathcal{C}$ misses some graph.

**Monadically dependent**, if every transduction of $\mathcal{C}$ misses some graph [Baldwin, Shelah '85]

Theorem (Downey, Fellows, Taylor '96)

FO model checking is AW*-complete on general graphs, thus unlikely FPT on independent classes.

Could it be that on every dependent class, it is FPT?
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Could it be that on every dependent class, it is FPT?
Classes with known tractable FO model checking

Theorem (B., Kim, Thomassé, Watrigant ’20)

**FO Model Checking** solvable in $f(|\varphi|, d)n$ on graphs with a $d$-sequence.
Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant ’21)

_Bounded twin-width classes are small._

Unifies and extends the same result for:

- $\sigma$-free permutations [Marcus, Tardos ’04]
- $K_t$-minor free graphs [Norine, Seymour, Thomas, Wollan ’06]
Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant ’21)

Bounded twin-width classes are small.

Subcubic graphs, interval graphs, triangle-free unit segment graphs have unbounded twin-width
Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant ’21)
*Bounded twin-width classes are small.*

Is the converse true for hereditary classes?

Conjecture (small conjecture)
*A hereditary class has bounded twin-width if and only if it is small.*
Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

**Theorem (B., Geniet, Kim, Thomassé, Watrigant ’21)**

*Bounded twin-width classes are small.*

Is the converse true for hereditary classes?

**Conjecture (small conjecture, refuted: B., Geniet, Tessera, Thomassé ’21+)***

*A hereditary class has bounded twin-width if and only if it is small.*
Recap of the main questions

- Can we efficiently approximate twin-width?
- Can we solve FO model checking on every dependent class?
- Is every hereditary small class of bounded twin-width?
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- Can we efficiently approximate twin-width?
- Can we solve FO model checking on every dependent class?
- Is every hereditary small class of bounded twin-width?

We answer all these questions positively in the case of ordered binary structures ≡ matrices on a finite alphabet.
Twin-width for unordered matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Encode a bipartite graph (or, if symmetric, a graph)
Twin-width for unordered matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Contraction of two columns (similar with two rows)
Twin-width for unordered matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & r & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & r & 0 & 1 & 1 & 0 \\
1 & 0 & r & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

The red degree is now the max number of \( r \) per row/column
Twin-width for unordered matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & \textcolor{red}{r} & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \textcolor{red}{r} & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & \textcolor{red}{r} & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

In the non-bipartite case, we force symmetric pairs of contractions
Twin-width for matrices

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

That was *not* the twin-width of *ordered* matrices
Twin-width for matrices

Let’s also record the columns disagreeing with the contraction.
Twin-width for matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

\[
\max_{\text{row, column}} \ (\text{number of red entries} + \text{red degree})
\]
Twin-width for matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

If you find it too clumsy, encode the linear order
Twin-width for matrices

\[
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 \\
2 & 3 & 3 & 2 & 2 & 0 & 1 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 3 & 2 & 0 & 1 & 1 & 0 \\
3 & 2 & 0 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

and we’re back to the unordered definition
Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{array}
\]

Maximum number of non-constant zones per column or row part = error value
Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are \textit{consecutive}

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Maximum number of non-constant zones per column or row part
\[=\text{error value}\]
### Partition viewpoint

**Matrix partition**: partitions of the row set and of the column set

**Matrix division**: same but all the parts are *consecutive*

![Matrix](image)

Maximum number of non-constant zones per column or row part

... until there are a single row part and column part
Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Twin-width as maximum error value of a contraction sequence
Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{ R^{(1)}, <^{(2)}, E^{(2)} \}$
($E^{(2)}$ becomes $E_1^{(2)}, \ldots, E_t^{(2)}$ for $[0, t]$-matrices)
Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{ R(1), <(2), E(2) \}$
($E(2)$ becomes $E_1^{(2)}, \ldots, E_t^{(2)}$ for $[0, t]$-matrices)

- $M \models R(x)$ iff $x$ is a row index
- $M \models x < y$ iff $x$ is a smaller index than $y$
- $M \models E(x, y)$ iff $M_{x,y} = 1$
Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{ R^{(1)}, <^{(2)}, E^{(2)} \}$
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tractable class: FO model checking solvable in time $f(\phi)|M|^{O(1)}$
Growth of classes

Our matrix classes are closed under taking submatrices

- Small class: \#n \times n matrices is 2^{O(n)}
- Subfactorial: ultimately, \#n \times n matrices < n!

No non-trivial automorphism in totally ordered structures, so no need for labels
Equivalences in the matrix language

Theorem
For every matrix class $\mathcal{M}$, the following are equivalent.

(i) $\mathcal{M}$ has bounded twin-width.

(ii) $\mathcal{M}$ has bounded grid rank. (division property)

(iii) $\mathcal{M}$ is pattern-avoiding.
      (not including any of 6 “permutation-universal” classes)

(iv) $\mathcal{M}$ is dependent.

(v) $\mathcal{M}$ is monadically dependent.

(vi) $\mathcal{M}$ has subfactorial growth.

(vii) $\mathcal{M}$ is small.

(viii) $\mathcal{M}$ is tractable. (only if $\text{FPT} \neq \text{AW}[\ast]$.)

(ix) $\mathcal{M}$ has no large rich division. (division property)
Roadmap

(i) bounded twin-width

(ix) no large rich division

(ii) bounded grid rank

[vi] subfactorial growth

(iii) pattern-avoiding

(vii) small

(iv) independent

(viii) intractable

(i) bounded twin-width

(viii) tractable

(iv) dependent

(v) monadically dependent

Tww I

Tww II

Tww I

if FPT $\neq$ AW[*]
Roadmap

(i) unbounded twin-width

(ii) unbounded grid rank

(iii) “permutation-universal”

(iv) independent

(v) monadically dependent

(vi) factorial growth

(vii) small

(viii) intractable

(ix) large rich division

T ww I if FPT $\neq$ AW[*]
Theorem
Let $\mathcal{C}$ be a hereditary class of ordered graphs. The following are equivalent.

(1) $\mathcal{C}$ has bounded twin-width.
(2) $\mathcal{C}$ is monadically dependent.
(3) $\mathcal{C}$ is dependent.
(4) $\mathcal{C}$ is small.
(5) $\mathcal{C}$ contains $2^{O(n)}$ ordered $n$-vertex graphs.
(6) $\mathcal{C}$ contains less than $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$ ordered $n$-vertex graphs, for some $n$.
(7) $\mathcal{C}$ does not include one of 25 hereditary ordered graph classes with unbounded twin-width.
(8) FO-model checking is fixed-parameter tractable on $\mathcal{C}$. 

Equivalences in the ordered graph language
\textit{k-Rich division}

Division
**k-Rich division**

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Division such that for each, say, column part $C$
Division such that for each, say, column part \( C \) no removal of \( k \) row parts
$k$-Rich division

Division such that for each, say, column part $C$ no removal of $k$ row parts leaves $C$ with less than $k$ distinct column vectors.
Fix an $2k(k + 1)$-rich division $D$, and assume there is a $k$-sequence $S$
Large rich division $\Rightarrow$ unbounded twin-width

Consider the first time a part of $S$ intersects 3 parts of $D$
Large rich division $\Rightarrow$ unbounded twin-width

There are at most $k$ other column parts intersecting $C'_b$ (red degree of $C_j$)
Large rich division $\Rightarrow$ unbounded twin-width

Each such part $C_z$ is non-vertical in at most $2k$ zones of $\mathcal{D}$
Large rich division $\Rightarrow$ unbounded twin-width

Thus removing $2k(k + 1)$ row parts of $D \rightarrow \leq k + 1$ distinct columns
No large rich division $\Rightarrow$ bounded twin-width

Build greedily a division where every part contradicts the richness

- can only be stopped by a large rich division
- turned into a contraction sequence as in Tww I
No large rich division $\Rightarrow$ bounded twin-width

Build greedily a division where every part contradicts the richness

- can only be stopped by a large rich division
- turned into a contraction sequence as in $\text{Tww I}$

$\Rightarrow$ approximation of twin-width for ordered binary structures

**Theorem**

*There is a fixed-parameter algorithm, which, given an ordered binary structure $G$ and a parameter $k$, either outputs*

- a $2^{O(k^4)}$-sequence of $G$, implying that $\text{tww}(G) = 2^{O(k^4)}$, or
- a $2k(k+1)$-rich division of $M(G)$, implying that $\text{tww}(G) > k$. 
Roadmap

$T_{ww} I$ if $\text{FPT} \not= \text{AW}[*]$

(i) unbounded twin-width

(ii) unbounded grid rank

(iii) “permutation-universal”

(iv) independent

(vi) factorial growth

(ix) large rich division

(viii) intractable
**$k$-rank division**

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$k$-by-$k$ division where every cell has rank at least $k$
Grid rank of $M = $ largest $k$ such that $M$ admits a $k$-rank division
Large rich division $\Rightarrow$ unbounded grid rank

Fix a large rich division $\mathcal{D}$
Large rich division $\Rightarrow$ unbounded grid rank

Red zones = large rank; Blue zones = first of its column to contain a particular row vector
Large rich division $\Rightarrow$ unbounded grid rank

Marcus-Tardos theorem applied to the colored zones $\rightarrow$ division $D'$
Large rich division $\Rightarrow$ unbounded grid rank

Coarser division $\mathcal{D}''$, 1 zone of $\mathcal{D}'' \equiv 2^k \times 2^k$ zones of $\mathcal{D}'$
Large rich division $\Rightarrow$ unbounded grid rank

A zone of $D''$ containing a red zone has large rank
Large rich division $\Rightarrow$ unbounded grid rank

Other zones have diagonals of blue zones
Large rich division $\Rightarrow$ unbounded grid rank

$2^k$ distinct row vectors in each zone of $D''$
Large rank division $\Rightarrow$ large rank Latin division

Latin rank division: high-rank zones are boxed (red) in a universal permutation pattern,
Large rank division $\Rightarrow$ large rank Latin division

...they are the usual suspects: diagonal, anti-diagonal, upper triangular, upper anti-triangular, and their *complements*
Large rank division $\Rightarrow$ large rank Latin division

...while every other subzones are constant.
Large rank division $\Rightarrow$ large rank Latin division

Reversible encoding of \[ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] by a $6 \times 6$ matrix
Large rank division $\Rightarrow$ large rank Latin division

Injection from $\mathcal{S}_n$ to $\mathcal{M}_{2n}$ $\rightarrow$ $|\mathcal{M}_n| \geq \lfloor \frac{n}{2} \rfloor !$
Roadmap

(i) unbounded twin-width

(ix) large rich division

(ii) unbounded grid rank

(vi) factorial growth

(iii) “permutation-universal”

(iv) independent

(viii) intractable

Tww I

if FPT \(\not=\) AW[\(*]\)
Further extractions in the rank Latin division

Submatrix agreeing on 1 of 16 patterns for the constant zones

\( \eta : \{-1, 1\}^2 \cup \{(0, 0)\} \rightarrow \{0, 1\} \) with \( \eta(0, 0) = 1 - \eta(1, 1) \)
Large rank Latin division $\Rightarrow$ permutation-universal

An example of a pattern with $\eta(x, y) = 0$ iff $x = y = 1$
Large rank Latin division $\Rightarrow$ permutation-universal

Another example
Large rank Latin division $\Rightarrow$ permutation-universal

Now injection from $\mathfrak{S}_n$ to $\mathcal{M}_n$, so $|\mathcal{M}_n| \geq n!$
Only 6 minimal permutation-universal classes
Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

Every hereditary class of ordered graphs have growth $2^{O(n)}$
or at least $n^{n/2+o(n)}$.

Solved:

- Bounded twin-width: growth is $2^{O(n)}$ (Tww II)
- Unbounded twin-width: $\geq n!$ ordered $(n, n)$-bipartite graphs
Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

Every hereditary class of ordered graphs have growth $2^{O(n)}$

or at least $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! = n^{n/2+o(n)}$

Solved:

- Bounded twin-width: growth is $2^{O(n)}$ (Tww II)
- Unbounded twin-width: $\geq n!$ ordered $(n, n)$-bipartite graphs

A bit more work to get the fine-grained bound
Roadmap

(i) bounded twin-width
(ii) bounded grid rank
(iii) pattern-avoiding
(iv) dependent
(v) monadically dependent
(vi) subfactorial growth
(vii) small
(viii) tractable
(ix) no large rich division

Tww I

if FPT $\neq$ AW[*]
Roadmap

(ix) no large rich division

(ii) bounded grid rank

(vi) subfactorial growth

(iii) pattern-avoiding

(vii) small

(iv) dependent

(iii) tractable

(v) monadically dependent

(i) bounded twin-width

Tww I def

if FPT \neq AW[*]

Thank you for your attention!