Algorithmic developments using twin-width

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Twin-width

tww(G): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have \textit{maximum red degree} at most $d$.

Maximum red degree $= 0$

\textbf{overall maximum red degree} $= 0$
tww(G): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have *maximum red degree* at most $d$.

Maximum red degree = 2
overall maximum red degree = 2
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Maximum red degree = 2
overall maximum red degree = 2
Twin-width

tww(\(G\)): Least integer \(d\) such that \(G\) admits a contraction sequence where all trigraphs have maximum red degree at most \(d\).

Maximum red degree = 2
overall maximum red degree = 2
Twin-width

tww(G): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have maximum red degree at most $d$.

Maximum red degree = 1
overall maximum red degree = 2
Twin-width

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Twin-width

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Partition viewpoint: $S = \mathcal{P}_n, \mathcal{P}_{n-1}, \ldots, \mathcal{P}_1$ with $\mathcal{P}_i$ a partition of $V(G)$ in $i$ parts
Theorem (B., Geniet, Kim, Thomassé, Watrigant ’20 & ’21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- Bounded rank-width or clique-width graphs,
- every hereditary proper subclass of permutation graphs,
- posets of bounded antichain size,
- unit interval graphs,
- $K_t$-minor free graphs,
- map graphs,
- subgraphs of d-dimensional grids,
- $K_t$-free unit $d$-dimensional ball graphs,
- $\Omega(\log n)$-subdivisions of all the $n$-vertex graphs,
- cubic expanders defined by iterative random 2-lifts from $K_4$,
- strong products of two bounded twin-width classes, one with bounded degree,
- first-order (FO) transductions of the above.
Given a $d$-sequence, one can solve any problem definable with a FO sentence $\varphi$ in time $f(d, \varphi)|V(G)|$. Special cases like $k$-INDEPENDENT SET in time $2^{O_d(k)}|V(G)|$.

$O_d(1)$-approximate MIN DOMINATING SET
How hard is computing twin-width?

Theorem (Bergé, B., Déprés '22)

*It is NP-complete to decide if the twin-width is at most 4.*
How hard is computing twin-width?

Theorem (Bergé, B., Déprés '22)

*It is NP-complete to decide if the twin-width is at most 4.*

Question

*Given a graph $G$ and an integer $d$, is it possible to either provide an $f(d)$-sequence of $G$ or correctly conclude that $\text{tww}(G) > d$, in time $g(d)|V(G)|^{O(1)}$ or $|V(G)|^{g(d)}$?*

Question

*Is twin-width at most $d$ polytime recognizable? (for $d \in \{2, 3\}$)*
Here with $k = 3$, it has every 3-permutation as subpermutation.
The 6 universal patterns of unbounded twin-width

**Theorem (B., Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk ’22)**

$\exists f \text{ s.t. all the adjacency matrices of a graph of twin-width } \geq f(k)\text{ contains a } k\text{-grid permutation submatrix or one of its 5 encodings}$

Semi-induced matching, antimatching, and 4 half-graphs or ladders
Twin-width win-win

Goal: compute FO-definable parameter $p$ in FPT time in $\mathcal{C}$.

Show that $\exists f$ non-decreasing, such that $\forall G \in \mathcal{C}$ an $f(p(G))$-sequence of $G$ can be computed in FPT time

- **Width $> f(k)$**: report $p(G) > k$
- **Width $\leq f(k)$**: use FO model checking algorithm

$\rightarrow \text{k-Biclique in visibility graphs of 1.5D terrains}$
$\rightarrow \text{k-Independent Set in visibility graphs of simple polygons}$

[B., Chakraborty, Kim, Köhler, Lopes, Thomassé '22]
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[B., Chakraborty, Kim, Köhler, Lopes, Thomassé ’22]
Visibility graphs of 1.5D terrains

Order along $x$-coordinates
Visibility graphs of 1.5D terrains

Order along $x$-coordinates

$k$-Biclique and $k$-Ladder are FPT in this class
Ordering along the boundary of the polygon
Ordering along the boundary of the polygon

\[ \begin{array}{c}
  a & b & c & d \\
  b' & c' & a & b \\
  c' & d & c & b'
\end{array} \]
Here we only need a decreasing pattern

By Ramsey's theorem, we can assume that the $\alpha_i$s and the $\beta_i$s both induce a clique.
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Geometric arguments

Quadrangle $\alpha_2\alpha_3\beta_3\beta_2$ is not self-crossing
Geometric arguments

Quadrangle $\alpha_2 \alpha_3 \beta_3 \beta_2$ has to be convex
Geometric arguments

Then $\alpha_2, \alpha_3, \beta_3, \beta_2$ induce $K_4$, a contradiction
Approximation algorithms

Theorem (B., Geniet, Kim, Thomassé, Watrigant ’21)
For every $d$, there is a $D$ such that every $n$-vertex graph with twin-width at most $d$ iteratively admits $\frac{n}{D}$ disjoint pairs that can be contracted in a $D$-sequence.
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Consequence: We can turn a $d$-sequence into a balanced $D$-sequence $S$, i.e., such that $\forall P_i \in S, \forall P \in P_i, |P| \leq D\frac{n}{i}$
Approximating \textbf{Max Independent Set}

In general graphs: an $n^{1-\varepsilon}$-approximation or $r(n)$-approximation in time $\exp\left(\frac{n^{1-\varepsilon}}{r(n)^{1+\varepsilon}}\right)$ are unlikely.
Approximating \textbf{Max Independent Set}

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$P_{\sqrt{n}}$
Approximating **Max Independent Set**

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\[ \mathcal{P}_{\sqrt{n}} \quad C_1 \]
\[ \quad C_2 \]
\[ \quad C_3 \]

$D + 1$-color the red graph of $G/\mathcal{P}_{\sqrt{n}}$ in polynomial time
Approximating **Max Independent Set**

In general graphs: an $n^{1-\varepsilon}$-approximation or $r(n)$-approximation in time $\exp(\frac{n^{1-\varepsilon}}{r(n)^{1+\varepsilon}})$ are unlikely

Solve MIS in $G[P_j]$ for every $P_j \in \mathcal{P}_{\sqrt{n}}$ in $2^{O(D\sqrt{n})}$ time
Approximating \textbf{Max Independent Set}

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Solve weighted MIS in $G/P_{\sqrt{n}}[C_i], \forall i \in [D+1]$ in $2^{O(\sqrt{n})}$ time
Approximating \textbf{Max Independent Set}

In general graphs: an $n^{1-\varepsilon}$-approximation or $r(n)$-approximation in time $\exp\left(\frac{n^{1-\varepsilon}}{r(n)^{1+\varepsilon}}\right)$ are unlikely

A heaviest such solution is a $(D + 1)$-approximation
Approximating MIS given a $d$-sequence

Theorem (Bergé, B., Déprés, Watrigant ’22+)
MIS can be $O_d(1)$-approximated in time $2^{O_d(\sqrt{n})}$.
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MIS can be $O_d(1)^{2^q-1}$-approximated in time $2^{O_{d,q}(n^{2^{-q}})}$, $\forall q \in \mathbb{N}$. 
Approximating MIS given a $d$-sequence

**Theorem (Bergé, B., Déprés, Watrigant ’22+)**

MIS *can be* $O_d(1)$-approximated *in time* $2^{O_d(\sqrt{n})}$.

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**Theorem (Bergé, B., Déprés, Watrigant ’22+)**

MIS *can be* $O_d(1)^{2^q-1}$-approximated *in time* $2^{O_d,q(n^2-q)}$, $\forall q \in \mathbb{N}$.

Setting $q := \log \frac{\varepsilon \log n}{O_d(1)}$

**Theorem (Bergé, B., Déprés, Watrigant ’22+)**

MIS *can be* $n^\varepsilon$-approximated *in polynomial time.*
COLORING, MAX INDUCED MATCHING

Similar results for these problems
Open questions

FPT/XP approximation of twin-width

Practical FPT algorithms for the problems on polygons/terrains

Better than $n^\varepsilon$-approximation for MIS given $O(1)$-sequences? (every such approximation would then self-improve)

unexpected use of the FO model checking algorithm on bounded tww (recent example with 3-terminal pairs Directed Multicut)
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Thank you for your attention!