Inapproximability of Diameter in super-linear time: Beyond the 5/3 ratio

Édouard Bonnet

ENS Lyon, LIP

March 16th, 2021, STACS
\[\forall k, \exists \epsilon > 0, \text{no classical algorithm solves } n\text{-var } k\text{-SAT in } (2 - \epsilon)^n\]

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

\[\text{SETH} \Rightarrow \text{ETH} \Rightarrow P \neq NP\]

- ETH and SETH are then mainly used for NP-hard problems
∀k, ∃ε > 0, no classical algorithm solves $n$-var $k$-SAT in $(2 - \varepsilon)^n$

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- In 2005, SETH is used for the first time for a problem in P

Orthogonal Vectors,
∀k, ∃ε > 0, no classical algorithm solves \( n \)-var \( k \)-SAT in \( (2 − \varepsilon)^n \)

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

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\text{SETH} \Rightarrow \text{ETH} \Rightarrow \text{P} \neq \text{NP}
\]

- ETH and SETH are then mainly used for NP-hard problems
- In 2005, SETH is used for the first time for a problem in \( \text{P} \)
- 2014-, dozens of papers show SETH-hardness of problems in \( \text{P} \)

Orthogonal Vectors, Diameter, Fréchet Distance, Edit Distance, Longest Common Subsequence, Furthest Pair, dynamic problems, problems from Machine Learning, Model Checking, Language Theory etc.
Reduction from \textbf{SAT} to a problem in P

2-Orthogonal Vectors (2-OV):
Are there two orthogonal vectors in a given set of $N$ 0, 1-vectors?

\begin{align*}
\text{problem } \textbf{SAT} \\
\text{instance } \phi \\
n \text{ variables}
\end{align*}
\begin{align*}
\rightarrow \\
\text{reduction} \\
\text{time } \leq 1.99^n
\end{align*}
\begin{align*}
\text{problem 2-OV} \\
\text{instance } \mathcal{V} \\
N = 2^{n/2} \text{ vectors}
\end{align*}
Reduction from $\text{SAT}$ to a problem in $\text{P}$

2-ORTHOGONAL VECTORS (2-OV):
Are there two orthogonal vectors in a given set of $N$ 0, 1-vectors?

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\begin{align*}
\text{problem } \text{SAT} & \quad \text{instance } \phi \\
& \quad n \text{ variables} \\
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\text{problem 2-OV} & \quad \text{instance } \mathcal{V} \\
& \quad N = 2^{n/2} \text{ vectors}
\end{align*}
\]

$\rightarrow$ Solving 2-OV in $N^{1.99}$ solves $\text{SAT}$ $1.99^n + 2^{\frac{1.99n}{2}}$, refuting SETH
arbitrary equipartition of $X$: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- $A$ of the red variables and
- $B$ of the blue variables

such that all the clauses are satisfied by $A$ or by $B$
**Sat → 2-Orthogonal Vectors [W05]**

arbitrary equipartition of $X$: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2} + 1}, x_{\frac{n}{2} + 2}, \ldots, x_n$

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**Sat → 2-Orthogonal Vectors [W05]**

arbitrary equipartition of $X$: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

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$A_1$ assigns red variables
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$A_1$ does not satisfy $C_1$
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$A_1$ satisfies $C_2$
Sat → 2-Orthogonal Vectors [W05]

arbitrary equipartition of \( X \): \( x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_n \)

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first vector \((1, 0, 1, 0, 0, 1, 1, 0, 1, 0)\)
Sat $\rightarrow$ **2-Orthogonal Vectors [W05]**

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</tbody>
</table>
Consequence for 2-OV

From a SAT-instance on $n$ variables and $m$ clauses, we created $N := 2^{\frac{n}{2}+1}$ vectors in dimension $d := m + 2$.
Consequence for 2-OV

From a \( SAT \)-instance on \( n \) variables and \( m \) clauses, we created \( N := 2^{\frac{n}{2} + 1} \) vectors in dimension \( d := m + 2 \)

An algorithm solving 2-OV in time \( 2^{o(d)} \cdot N^{2-\varepsilon} \) would solve \( SAT \) in \( 2^{o(m)} \cdot 2^{n(1-\varepsilon/2)} \) → breaking SETH
Consequence for 2-OV

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An algorithm solving 2-OV in time \( 2^{o(d)} N^{2-\varepsilon} \) would solve SAT in \( 2^{o(m)} 2^{n(1-\varepsilon/2)} \to \) breaking SETH

Most useful consequence here:
\( N^{2-o(1)} \)-time is required even if \( d = \log^{O(1)} N \)
Same for \( k \)-OV and \( N^{k-o(1)} \)-time
Diameter

diam(G) = largest distance between a pair of vertices of G

$\overset{u}{\bullet} \overset{\text{longest}_{u,v} \ \text{shortestPath}(u, v)}{\longrightarrow} \overset{v}{\bullet}$

- In weighted graphs, no better known than APSP in $O(n^3)$
- In unweighted graphs, solvable in $\tilde{O}(n^\omega)$
- In unweighted sparse ($m = \Theta(n)$) graphs, solvable in $O(n^2)$
- $\frac{3}{2}$-approximable in $\tilde{O}(m^{1.5})$
- In sparse graphs, $\frac{3}{2}$-approximable in $\tilde{O}(n^{1.5})$
Diameter

diam(G) = largest distance between a pair of vertices of \( G \)

\[
\begin{array}{c}
\text{u} \\
\bullet \\
\text{v}
\end{array}
\]

\( \text{longest}_{u,v} \text{ shortestPath}(u, v) ? \)

- In weighted graphs, no better known than APSP in \( O(n^3) \)
- In unweighted graphs, solvable in \( \tilde{O}(n^\omega) \)
- **In unweighted sparse graphs**, solvable in \( O(n^2) \)
- \( \frac{3}{2} \)-approximable in \( \tilde{O}(m^{1.5}) \)
- **In sparse graphs**, \( \frac{3}{2} \)-approximable in \( \tilde{O}(n^{1.5}) \)

Linear reduction from **Orthogonal Vectors**: no \( n^{1.99} \) algorithm even to \( (\frac{3}{2} - \varepsilon) \)-approximate Diameter on unweighted sparse instances, assuming the SETH.
2-Orthogonal Vectors $\rightarrow$ Diameter [RV13]

So far, all the pairs but of $A \times B$ are at distance $\leq 2$
2-Orthogonal Vectors $\rightarrow$ Diameter [RV13]

We put an edge between vector $v$ and index $i$ iff $v[i] = 1$.
A pair \((a_4, b_2)\) is at distance 2 \(\iff \langle a_4, b_2 \rangle \neq 0\)
2-Orthogonal Vectors $\rightarrow$ Diameter [RV13]

$\text{diam}(G) = 3 \iff \exists (a_i, b_j) \text{ at distance 3} \iff \text{orthogonal pair}$
2-Orthogonal Vectors $\rightarrow$ Diameter [RV13]

If no orthogonal pair, $\text{diam}(G) = 2$
Time-approximation trade-offs of directed weighted Diameter

<table>
<thead>
<tr>
<th>Approximation Factor</th>
<th>Runtime Exponent</th>
</tr>
</thead>
<tbody>
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<td>4/3</td>
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</tbody>
</table>

- RV13
- CLRSTV14
- Li20
- BRSWW18
Time-approximation trade-offs of directed weighted DiAMETER

![Graph showing trade-offs between runtime exponent and approximation factor.]

- RV13
- CLRSTV14
- RV13
- Li20
- BRSVW18
- Our result
Our result

Theorem
Approximating weighted directed \texttt{Diameter} within factor better than \(\frac{7}{4}\) requires time \(n^{\frac{4}{3} - o(1)}\), unless SETH fails.
Our result

Theorem

Approximating weighted directed \textsc{Diameter} within factor better than \( \frac{7}{4} \) requires time \( n^{\frac{4}{3} - o(1)} \), unless SETH fails.

Plan: hardness of 4 vs 7 \textsc{Diameter} from \( N \)-vector 4-OV to \( O(N^3) \)-vertex \( \tilde{O}(N^3) \)-arc \textsc{Diameter}-instances.
Theorem

Approximating weighted directed $\text{Diameter}$ within factor better than $\frac{7}{4}$ requires time $n^{\frac{4}{3}-o(1)}$, unless $SETH$ fails.

Plan: hardness of 4 vs 7 $\text{Diameter}$ from $N$-vector 4-OV to $O(N^3)$-vertex $\tilde{O}(N^3)$-arc $\text{Diameter}$-instances.

2 vs 3 hardness from 2-OV [RV13]
3 vs 5 hardness from 3-OV [Li20]
Will define the six sets ABC, AB, AD_Y, AD_X, DC, DCB later and the variable edge sets
Global structure

(1) Yes(4-OV) ⇒ ∃(u, v) ∈ ABC × DCB with d(u, v) = 7
(2) No(4-OV) ⇒ ∀u, v, d(u, v) ≤ 4
(2) is satisfied except for $ABC \times AD_Y$, $ABC \times DC$, $ABC \times DCB$, $AB \times DC$, $AB \times DCB$, $AD_Y \times DCB$
Dalirrooyfard and Wein observed that these unweighted arcs suffice.
Vertex sets

$S = A = B = C = D$ is the input set of 0, 1-vectors in $\{0, 1\}^\ell$

Vertices:
- $(a, b, c) \in ABC$ for every $(a, b, c) \in A \times B \times C$
- $(d, c, b) \in DCB$ for every $(d, c, b) \in D \times C \times B$
Vertex sets

\[ S = A = B = C = D \] is the input set of 0, 1-vectors in \( \{0, 1\}^\ell \)

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- \((d, c, b) \in DCB\) for every \((d, c, b) \in D \times C \times B\)
- \((a, b, i, j, k) \in AB\) for every \((a, b) \in A \times B\), \(i, j, k \in [\ell]\) and \(a[i] = a[j] = a[k] = 1\) and \(\text{maj}(b[i], b[j], b[k]) = 1\)
- same for DC
Vertex sets

\[ S = A = B = C = D \] is the input set of 0, 1-vectors in \( \{0, 1\}^\ell \)

Vertices:

- \((a, b, c)\) \(\in\) ABC for every \((a, b, c)\) \(\in\) \(A \times B \times C\)
- \((d, c, b)\) \(\in\) DCB for every \((d, c, b)\) \(\in\) \(D \times C \times B\)
- \((a, b, i, j, k)\) \(\in\) AB for every \((a, b)\) \(\in\) \(A \times B\), \(i, j, k \in [\ell]\) and \(a[i] = a[j] = a[k] = 1\) and \(\text{maj}(b[i], b[j], b[k]) = 1\)
- same for DC
- \((a, d, i, j, k)_Y\) \(\in\) \(AD_Y\) for every \((a, b)\) \(\in\) \(A \times D\), \(i, j, k \in [\ell]\) and \(a[i] = a[j] = a[k] = d[i] = d[j] = d[k] = 1\)
- \((a, d, i, j, k)_X\) \(\in\) \(AD_X\) for every \((a, b)\) \(\in\) \(A \times D\), \(i, j, k \in [\ell]\) and \(a[i], a[j], a[k], d[i], d[j], d[k]\) equal 1 but at most 1.
Edge sets

\[ (a, b, i, j, k)_{\gamma} \quad (a', d, i'', j'', k'')_{\gamma} \]

\[ (a, d, i, j, k)_{\gamma} \quad (a', d, i'', j'', k'')_{\gamma} \]

\[ d[i] = d[j] = d[k] = 1 \]

\[ a[i] = a[j] = a[k] = 1 \]

\[ \exists h \in \{i, j, k\}, c[h] = b[h] = 1 \]

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\[ a[i] = a[j] = a[k] = 1 \]

\[ d[i] = d[j] = d[k] = 1 \]

\[ \text{ABC} \]

\[ \text{DCB} \]

\[ (a, b, c) \]

\[ (a', b', i', j', k') \]

\[ (a', d, i', j', k') \]

\[ \text{Regular edges,} \]

\[ \text{index-switching edges,} \]

\[ \text{skew edges} \]
**Edge sets**

\[
(a, b, i, j, k) \quad (a', d, i', j', k')
\]

\[
(a, d, i, j, k)_Y \quad (a', d, i'', j'', k'')_Y
\]

\[
a[i] = a[j] = a[k] = 1
\]

\[
d[i] = d[j] = d[k] = 1
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\]

Regular edges, **index-switching edges**,
Edge sets

\[ a[i] = a[j] = a[k] = 1 \]
\[ d[i] = d[j] = d[k] = 1 \]
\[ (a, d, i, j, k) \] \( \gamma \) \( (a', d, i'', j'', k'') \) \( \gamma \)

\( a[i] = a[j] = a[k] = 1 \)
\( \text{maj}(b[i], b[j], b[k]) = 1 \)
\( (a, b, i, j, k) \)
\( (a, b, i', j', k') \)

\( \exists h \in \{i, j, k\}, c[h] = b[h] = 1 \)
\( ABC \)
\( (a, b, c) \)

\( (a', d, i, j, k) \) \( \chi \)
\( \geq 5 \text{ of } a'[i], a'[j], a'[k], \)
\( d[i], d[j], d[k] \text{ are 1} \)

\( d[i] = d[j] = d[k] = 1 \)
\( \text{maj}(c[i], c[j], c[k]) = 1 \)
\( DC \)
\( (d, c, i, j, k) \)

\( \exists h \in \{i, j, k\}, b[h] = c[h] = 1 \)
\( DCB \)
\( (d, c, b) \)

Regular edges, index-switching edges, skew edges
No orthogonal quadruple in $S \Rightarrow \forall u, v, \ d(u, v) \leq 4$

**Important fact:** For every $a, b, c, d \in S$, $\exists i = \text{ind}(a, b, c, d) \in [\ell]$, $a[i] = b[i] = c[i] = d[i] = 1$

$$ABC \times DCB: \ i = \text{ind}(a, b, c, d), \ j = \text{ind}(a, b, c', d), \ k = \text{ind}(a, b', c', d)$$

$$(a, b, c) \rightarrow (a, b, i, j, k) \rightarrow (a, d, i, j, k) \rightarrow (d, c', i, j, k) \rightarrow (d, c', b')$$
No orthogonal quadruple in $S \Rightarrow \forall u, v, \ d(u, v) \leq 4$

\[
\begin{align*}
\text{a}[i] = a[j] = a[k] &= 1 \\
d[i] = d[j] = d[k] &= 1 \\
(a, d, i, j, k)_Y &\ (a', d, i'', j'', k'')_Y \\
(a, b, i', j', k') &\ (d, c, i'', j'', k''') \\
\end{align*}
\]

\[\exists h \in \{i, j, k\}, \text{c}[h] = b[h] = 1\]

\[
\begin{align*}
\text{a}[i] = a[j] = a[k] &= 1 \\
maj(b[i], b[j], b[k]) &= 1 \\
\exists h \in \{i, j, k\}, \text{c}[h] = b[h] = 1 \\
\end{align*}
\]

\[\exists h \in \{i, j, k\}, \text{c}[h] = b[h] = 1\]

\[\text{ABC} \times \text{DC}: \ i = \text{ind}(a, b, c, d), j = \text{ind}(a, b, c', d), k = \text{ind}(a, c', d)\]

\[(a, b, c) \rightarrow (a, b, i, j, k) \rightarrow (a, d, i, j, k)_Y \rightarrow (d, c', i, j, k) \rightarrow (d, c', i', j', k')\]
No orthogonal quadruple in $S \Rightarrow \forall u, \nu, \ d(u, \nu) \leq 4$

ABC $\times$ AD$_Y$: $i = \text{ind}(a, b, c, d), j = \text{ind}(a, a', b, d), k = \text{ind}(a, a', d)$

$(a, b, c) \rightarrow (a, b, i, j, k) \rightarrow (a, d, i, j, k)_Y \rightarrow (a', d, i, j, k)_X \rightarrow (a', d, i', j', k')_Y$
No orthogonal quadruple in \( S \Rightarrow \forall u, v, \ d(u, v) \leq 4 \)

\[
\begin{align*}
a[i] &= a[j] = a[k] = 1 \\
d[i] &= d[j] = d[k] = 1 \\
(a, d, i, j, k)_\gamma &\ (a', d, i'', j'', k'')_\gamma \\
(a', d, i', j', k')_x &\ (d, c, i'', j'', k''')_x \\
\text{maj}(b[i], b[j], b[k]) = 1 &\ \exists h \in \{i, j, k\}, \ c[h] = b[h] = 1 \\
\text{maj}(b[i], b[j], b[k]) = 1 &\ \exists h \in \{i, j, k\}, \ b[h] = c[h] = 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{AB} &\ \text{DC} : \ i = \text{ind}(a, b, c, d) \\
(a, b, i', j', k') &\rightarrow (a, b, i, i, i) \rightarrow (a, d, i, i, i)_\gamma \rightarrow (d, c, i, i, i) \rightarrow (d, c, i'', j'', k'') \\
\end{align*}
\]
a, b, c, d orthogonal $\Rightarrow d((a, b, c), (d, c, b)) = 7$

No path of length at most 6 via $u$ or $v$
a, b, c, d orthogonal \Rightarrow d((a, b, c), (d, c, b)) = 7

\begin{align*}
a[i] &= a[j] = a[k] = 1 \\
d[i] &= d[j] = d[k] = 1 \\
(a, d, i, j, k)_\gamma &= (a', d, i'', j'', k'')_\gamma \\
an\geq 5 of a'[i], a'[j], a'[k], d[i], d[j], d[k] are 1
\end{align*}

\begin{align*}
\exists h \in \{i, j, k\}, c[h] &= b[h] = 1 \\
\exists h \in \{i, j, k\}, b[h] &= c[h] = 1
\end{align*}

Case (a): no skew edge used
$a, b, c, d$ orthogonal $\Rightarrow d((a, b, c), (d, c, b)) = 7$

Implies $(a, b, i, j, k) \in AB$ and $(d, c, i, j, k) \in DC$ on the path, thus $\exists h \in \{i, j, k\}$ $a[h] = b[h] = c[h] = d[h] = 1$, a contradiction
\(a, b, c, d\) orthogonal \(\Rightarrow d((a, b, c), (d, c, b)) = 7\)

\[\begin{align*}
a[i] &= a[j] = a[k] = 1 \\
d[i] &= d[j] = d[k] = 1
\end{align*}\]

Case (b): at least one skew edge used
\((a, b, c) \rightarrow (a, b, i, j, k) \rightarrow (a, d, i, j, k)_Y\) (or symmetric), hence contradiction
Recent developments for directed unweighted Diameter

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<tr>
<td>$\frac{6}{5}$</td>
<td>$6/5$</td>
</tr>
<tr>
<td>$\frac{k+1}{k}$</td>
<td>$k+1/k$</td>
</tr>
</tbody>
</table>

- RV13
- CLRSTV14
- Li20
- Li21
- DW21
Recent developments for undirected unweighted \textbf{Diameter}

\begin{align*}
    k + 1 \leq k &< 2k - 1 \\
    \frac{k+1}{k} &< 1 \\
    6/5 &< 5/4 \\
    3/2 &< 4/3 \\
    \frac{5}{3} &< \frac{7}{4} \\
    2 - \frac{1}{2k-1} &< 2
\end{align*}