Fine-Grained Complexity in P

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LIP, ENS Lyon

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Reductions

Two interpretations

Positive: we can solve Π in \( t(n) + f(r(n)) \) with \( f \) is the time for \( Π' \)

Negative: \( Π' \) cannot be solved in \( f(n) \) since we know/assume that Π is not solvable in \( t(n) + f(r(n)) \)
Reductions

Two interpretations

Positive: we can solve $\Pi$ in $t(n) + f(r(n))$ with $f$ is the time for $\Pi'$

Negative: $\Pi'$ cannot be solved in $f(n)$ since we know/assume that $\Pi$ is not solvable in $t(n) + f(r(n))$
Reductions and fine-grained complexity

Complexity with the classes \( \text{TIME}(f(n)) \) and reference problems

Two interpretations

Positive: we can solve \( \Pi \) in \( t(n) + f(r(n)) \) with \( f \) is the time for \( \Pi' \)

Negative: \( \Pi' \) cannot be solved in \( f(n) \)
since we know/assume that \( \Pi \) is not solvable in \( t(n) + f(r(n)) \)
The three main hypotheses

**Strong Exponential Time Hypothesis (SETH):** \( \forall \varepsilon > 0, \exists \ k \ \text{s.t.} \ \ k\text{-SAT} \ \text{is not in time} \ 2^{(1-\varepsilon)n} \).
The three main hypotheses

**Strong Exponential Time Hypothesis (SETH):** $\forall \varepsilon > 0$, $\exists k$ s.t. $k$-SAT is not in time $2^{(1-\varepsilon)n}$ by a classical (randomized) algorithm.
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**3-SUM Hypothesis:** Finding $x, y, z$ such that $x + y + z = 0$ in a list of $n$ integers of $[-n^4, n^4]$ is not in time $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$. 

**All-Pairs Shortest-Path (APSP) Hypothesis:** $\exists c$, APSP with edge weights in $[-n^c, n^c]$ is not solvable in time $O(n^{3-\varepsilon})$ for $\varepsilon > 0$. 

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The three main hypotheses

**SETH:** \( \text{SAT} \) is not solvable in \( 1.99^n \).

- \( k\text{-SAT} \) is solvable in \( 2^{\left(1-\Theta\left(\frac{1}{k}\right)\right)n} \)
- \( \text{SAT} \) is solvable \( 2^{\left(1-\Theta\left(\frac{1}{\log m/n}\right)\right)n} \)

**3-SUM Hypothesis:** 3-SUM is not solvable in \( n^{1.99} \)

- Solvable in \( n^2 \frac{\log \log n}{\log^2 n}^{O(1)} \) even with real inputs
- Linear decision tree with depth \( O(n \log^2 n) \)

**APSP Hypothesis:** APSP is not solvable in \( n^{2.99} \)

- Solvable in cubic time by Floyd-Warshall algorithm
- Improved to \( n^3/2^{O(\sqrt{\log n})} \)
In 1999, Impagliazzo and Paturi introduce ETH\(^1\) and mention a stronger version of it in their conclusion

\[
\text{SETH} \implies \text{ETH} \implies P \neq NP
\]

- ETH and SETH are then mainly used for NP-hard problems

\(^1\exists \delta > 0, 3\text{-SAT cannot be solved in } 2^{\delta n}\)
SETH

In 1999, Impagliazzo and Paturi introduce ETH\(^1\) and mention a stronger version of it in their conclusion

\[ \text{SETH} \Rightarrow \text{ETH} \Rightarrow P \neq NP \]

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Orthogonal Vectors,
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- ETH and SETH are then mainly used for NP-hard problems
- In 2005, SETH is used for the first time for a problem in P
- 2014-\(^\), dozens of papers show SETH-hardness of problems in P

**Orthogonal Vectors, Diameter, Fréchet Distance, Edit Distance, Longest Common Subsequence, Furthest Pair, dynamic problems, problems from Machine Learning, Model Checking, Language Theory etc.**

\(^1\exists \delta > 0, 3\text{-Sat cannot be solved in } 2^{\delta n}\)
Psychological barrier?

If we treat $k$-$\text{SAT} \rightarrow \text{OV}$ as an outlier, why 15 years between defining SETH and using it in P?
Psychological barrier?

A: If you know too much (P, NP, NP-completeness), you might disregard a reduction from a hard problem to an easy one

B: If you know less (summing and composing functions)

\[
\text{problem SAT} \quad \text{instance } \phi \\
\text{size } n \\
\text{reduction time } \leq 1.99^n \\
\text{problem OV} \quad \text{instance } \forall \\
\text{size } r(n) = 2^{n/2}
\]
Psychological barrier?

A: If you know too much (P, NP, NP-completeness), you might disregard a reduction from a hard problem to an easy one.

B: If you know less (summing and composing functions)

<table>
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<tr>
<th>Problem ( SAT )</th>
<th>Reduction time ( \leq 1.99^n )</th>
<th>Problem ( OV )</th>
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<td>Instance ( \phi )</td>
<td>size ( n )</td>
<td>instance ( \mathcal{V} )</td>
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B: ”Both the time and the blow-up can be exponential, great.”
**SAT → ORTHOGONAL VECTORS**

arbitrary equipartition of $X$: $x_1, x_2, \ldots, x_{n/2}, x_{n/2}+1, x_{n/2}+2, \ldots, x_n$

Find an assignment

- $A$ of the red variables and
- $B$ of the blue variables

such that all the clauses are satisfied by $A$ or by $B$
SAT $\rightarrow$ ORTHOGONAL VECTORS

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$$

$A_1$ assigns red variables
**Sat → Orthogonal Vectors**

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$A_1$ does not satisfy $C_1$
SAT $\rightarrow$ Orthogonal Vectors

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$A_1$ satisfies $C_2$
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first vector $(1, 0, 1, 0, 0, 1, 1, 0, 1, 0)$
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Sat → Orthogonal Vectors

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</table>
Consequence for OV under the SETH

From a $\text{SAT}$-instance on $n$ variables and $m$ clauses, we created $N := 2^{\frac{n}{2} + 1}$ vectors in dimension $d := m + 2$
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From a SAT-instance on $n$ variables and $m$ clauses, we created $N := 2^{\frac{n}{2} + 1}$ vectors in dimension $d := m + 2$

An algorithm solving OV in time $2^{o(d)} N^{2 - \varepsilon}$ would solve SAT in $2^{o(m)} 2^n(1 - \varepsilon/2) \rightarrow$ breaking SETH

Sharp contrast with the simple algorithms in $O(N^2d)$ and $O(2^d N)$
**Diameter**

\[ \text{diam}(G) = \text{largest distance between a pair of vertices of } G \]

\[ \begin{array}{ccc} \text{longest}_{u,v} \text{ shortestPath}(u, v) \end{array} \]

- In weighted graphs, nothing known better than APSP
- In unweighted graphs, solvable in \( \tilde{O}(n^\omega) \)
- In unweighted sparse \((m = \Theta(n))\) graphs, solvable in \(O(n^2)\)
- \(\frac{3}{2}\)-approximable in \(\tilde{O}(m^{1.5})\)
- In sparse graphs, \(\frac{3}{2}\)-approximable in \(\tilde{O}(n^{1.5})\)
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$\text{longest}_{u,v} \text{ shortestPath}(u, v) ?$

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- **In sparse graphs**, $\frac{3}{2}$-approximable in $\tilde{O}(n^{1.5})$

Linear reduction from **Orthogonal Vectors**:

no $n^{1.99}$ algorithm even to $(\frac{3}{2} - \varepsilon)$-approximate Diameter on unweighted sparse instances, assuming the SETH.
So far, all the pairs but of $A \times B$ are at distance $\leq 2$. 

**Orthogonal Vectors → Diameter**
Orthogonal Vectors $\rightarrow$ Diameter

We put an edge between vector $v$ and index $i$ iff $v[i] = 1$.
Orthogonal Vectors → Diameter

A pair \((a_4, b_2)\) is at distance 2 ⇔ \(\langle a_4, b_2 \rangle \neq 0\)
Orthogonal Vectors → Diameter

$diam(G) = 3 \iff \exists (a_i, b_j) \text{ at distance 3} \iff \text{orthogonal pair}$
Orthogonal Vectors $\rightarrow$ Diameter

If no orthogonal pair, $\text{diam}(G) = 2$
3-SUM Hardness

Introduced in 1995 by Gajentaan and Overmars to explain why some geometric problems require quadratic time

- are there three aligned points?
- are there three lines meeting at a point? (same by duality)
- is there a hole in a union of triangles?
- computing the area of a union of triangles
- is a rectangle covered by a set of infinite strips?
- Line separator of a non-intersecting axis-parallel segments?
- motion planning problems
- visibility problems
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the quadratic algorithm is not easy
3-SUM $\rightarrow$ 3 COLLINEAR POINTS

Each integer $x$ is mapped to the point $(x, x^3)$
3-SUM $\rightarrow$ 3 Collinear Points

Each integer $x$ is mapped to the point $(x, x^3)$

If $a$, $b$, $c$ are pairwise distinct

$$a + b + c = 0 \iff (a, a^3), (b, b^3), (c, c^3) \text{ are aligned}$$
Convenient 3-points-on-a-line 3-SUM-hard variant

\[ B \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[-C/2 \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ A \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]
Convenient 3-points-on-a-line 3-SUM-hard variant

\[
\begin{align*}
B & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
-C/2 & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
A & \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\end{align*}
\]
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Points become tiny holes between horizontal segments
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Points become tiny holes between horizontal segments
SEGMENTS SEPARATION → COVERED RECTANGLE

Rotation + Duality! Points ↔ Lines & Vertical Segments ↔ Strips
Segments Separation $\rightarrow$ Covered Rectangle

Rotation + Duality! Points $\leftrightarrow$ Lines & Vertical Segments $\leftrightarrow$ Strips

Point not on any strip $\Leftrightarrow$ Line separating the segments
APSP Hardness

This hypothesis has emerged more recently, introduced by Ryan Williams and Virginia Vassilevska Williams in 2010

APSP is in time $n^{3-\varepsilon}$ iff so is one of:

- finding a triangle with negative weight
- finding the radius of a weighted graph
- does a given matrix represent a metric?
- finding a shortest cycle in a graph with non-negative weights
- $(\min, +)$ matrix multiplication
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Let us recall that unweighted APSP can be solved in $n^\omega$. 
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**The hypothesis of weighted problems**

Let us recall that unweighted APSP can be solved in $n^\omega$

**Tree Edit Distance** in truly subcubic time is APSP-hard.
APNT → Negative Triangle: a Turing reduction

All-Pairs Negative Triangle (APNT):
Given a tripartite graph on \((A, B, C)\), is there, for every pair \(b \in B, c \in C\), a vertex \(a \in A\) such that \(abc\) is a negative triangle?

One can show that APSP and APNT are equivalent
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\[ B \quad A \quad C \]

Arbitrary partitions into \(t = \frac{n}{3}\) groups of size \(\frac{n}{t} = \frac{n}{3}\). 

---

**Diagram:**

- **A**: The set of vertices labeled as \(A\).
- **B**: The set of vertices labeled as \(B\).
- **C**: The set of vertices labeled as \(C\).
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For each triple of classes, call **Negative Triangle**
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Write down that the pair \(bc\) is satisfied by \(a\) and remove \(bc\)
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Continue with the same triple of classes while possible
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Report if all the pairs of \(B \times C\) were satisfied
**APNT → Negative Triangle**: a Turing reduction

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Number of calls to **Negative Triangle**: \(\leq n^2 + t^3 = O(n^2)\)
All-Pairs Negative Triangle (APNT):
Given a tripartite graph on \((A, B, C)\), is there, for every pair 
\(b \in B, c \in C\), a vertex \(a \in A\) such that \(abc\) is a negative triangle?

Size of the subinstances: \(3n/t = O(n^{1/3})\)
Are they pairwise incomparable?
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Can we only use SETH by designing fine-grained reductions

- from $k$-SAT to 3-SUM
- from $k$-SAT to APSP?
Non-deterministic Strong Exponential Time Hypothesis

\[ k \text{-Taut} : \text{Are all the assignments of a } k \text{-DNF formula satisfying?} \]

\[ \text{NSETH: } \forall \varepsilon > 0, \exists k, k \text{-Taut is not in NTIME}(2^{(1 - \varepsilon)n}) \].

\(\Rightarrow\) NSETH would imply non-trivial circuit lower bounds
\(\Rightarrow\) NSETH is false if randomization is allowed
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3-SUM in truly subquadratic co-nondeterministic time

3-SUM is in coNTIME($\tilde{O}(n^{1.5})$)
3-SUM in truly subquadratic co-nondeterministic time

3-SUM is in coNTIME(\(\tilde{O}(n^{1.5})\))

Certificate for non-existence of a triple summing to 0:

- a prime \(p\) among the first \(n^{1.5}\) primes \(\mathbb{P}_{n^{1.5}}\),
- an integer \(t = \tilde{O}(n^{1.5})\), and
- a set \(S\) of \(t\) triples all summing to 0 modulo \(p\) but not to 0
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Why does such a certificate exist?

\[
|\{(a_i, a_j, a_k, p) \mid a_i + a_j + a_k = 0 \mod p\}| \leq n^3 \log(3n^c) = \tilde{O}(n^3)
\]

\[
\exists p \in \mathbb{P}_{n^{1.5}}, \ |\{(a_i, a_j, a_k) \mid x + y + z = 0 \mod p\}| = \tilde{O}(n^{1.5})
\]
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Given \( (p, t, S) \), we check that:
- all triples of \( S \) sum to a non-zero value multiple of \( p \)
- expand \( (\sum_{i} x^{a_i} \mod p)^3 \) with FFT
- check that the coefficients of \( x^0, x^p, x^{2p} \) sum to \( t \)
Consequences for the unification

3-SUM is in coNTIME($\tilde{O}(n^{1.5})$)

APSP is in coNTIME($\tilde{O}(n^{2+\frac{6+\omega}{9}})$)

No known implication, the dashed ones are ruled out under NSETH
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APSP is in coNTIME($\tilde{O}(n^{2+6+\omega \over 9})$)

A fine-grained **deterministic** reduction from \(k\)-\textit{SAT} to either of these problems would break NSETH

No known implication, the dashed ones are ruled out under NSETH
Things I did not mention

Log shavings and friends

- **Reductions from Circuit Sat** to consolidate a lower bound
- **2 hypotheses implying** SETH, 3-SUMH, and APSPH!
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Thanks for your attention!