Grundy Coloring & friends, Half-Graphs, Bicliques

Pierre Aboulker, Édouard Bonnet, Eun Jung Kim, and Florian Sikora

ENS Lyon, LIP

STACS 2020, Montpellier, March 11th
Grundy coloring

The worst way of reasonably coloring a graph.
Grundy coloring

The **worst way** of reasonably coloring a graph.

- Order the vertices $v_1, v_2, \ldots, v_n$ to **maximize** #colors used by the greedy coloring.
- Greedy coloring: $v_i$ gets the first color not appearing in its neighborhood.
Grundy coloring

The worst way of *reasonably* coloring a graph.

- Order the vertices \( v_1, v_2, \ldots, v_n \) to *maximize* #colors used by the *greedy* coloring.
- Greedy coloring: \( v_i \) gets the first color not appearing in its neighborhood.
Grundy coloring

The worst way of reasonably coloring a graph.

- Order the vertices $v_1, v_2, \ldots, v_n$ to maximize \#colors used by the greedy coloring.
- Greedy coloring: $v_i$ gets the first color not appearing in its neighborhood.
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Do you see a Grundy coloring reaching color 6?
Witness = induced subgraph having the same Grundy number.
\( k \)-witness = induced subgraph having Grundy number at least \( k \).
A brief History of Grundy colorings

- 1939: Studied in directed acyclic graphs by Grundy.
- 1979: Kristen and Selkow defines the Grundy number $\Gamma(G)$.
- 1983: Simmons defines the ochromatic$^1$ number $\chi^o(G)$.
- 1987: Erdős et al. prove that $\chi^o(G) = \Gamma(G)$.

---

$^1$maximum number of colors used among all vertex-orderings, with the following rules (1) proper coloring, and (2) minimizing the number of colors for that ordering.
Algorithmic motivations

- $\Gamma(G)$ upper bounds the number of colors used by any greedy heuristic for \textsc{Min Coloring}.
- $\Gamma(G) \leq C\chi(G)$ on some classes of graphs gives a $C$-approximation for \textsc{Min Coloring}.
- Online coloring.
- See Sampaio’s and Gastineau’s PhD theses for further motivations.
How many vertices (at most) do we need to achieve color $k$?
How many vertices (at most) do we need to achieve color $k$?

4
How many vertices (at most) do we need to achieve color $k$?
How many vertices (at most) do we need to achieve color $k$?
How many vertices (at most) do we need to achieve color $k$?
How many vertices (at most) do we need to achieve color $k$?

A minimal witness is of size at most $2^{k-1}$.

Theorem (Zaker '06)

The Grundy number can be computed in $f(k)n^{2^{k-1}}$. 
Main question

Can $\Gamma(G) = k$ be decided in $f(k)n^{O(1)}$? At least in $f(k)n^k^{O(1)}$?
Main contribution

Can $\Gamma(G) = k$ be decided in $f(k)n^{O(1)}$? At least in $f(k)n^{k^{O(1)}}$?

No and no!

Theorem

Under the ETH, Grundy Coloring is not solvable in $f(k)n^{2^{o(k)}}$. Also shows $W[1]$-hardness
Main contribution

Can $\Gamma(G) = k$ be decided in $f(k) n^{O(1)}$? At least in $f(k) n^{k^{O(1)}}$?

No and no!

**Theorem**

*Under the ETH, Grundy Coloring is not solvable in $f(k) n^{2^{o(k)}}$.*

Also shows W[1]-hardness

Main ingredients:

- Constant-length "paths" of half-graphs (propagation).
- Truncated binomial trees (verification gadget).

We will now see what those things are
First thoughts on the parameterized reduction

- Reduce from a hard parameterized problem (e.g. \( k\text{-MULTICOLORED CLIQUE} \)) and keep the parameter small.
First thoughts on the parameterized reduction

- Reduce from a hard parameterized problem (e.g. \(k\text{-MULTICOLORED CLIQUE}\)) and keep the parameter small.
- A typical input \(G\) has other \(k\)-witnesses than \(k\)-cliques.
- Need to "dilate" \(G\).
First thoughts on the parameterized reduction

- Reduce from a hard parameterized problem (e.g. \textit{k-Multicolored Clique}) and keep the parameter small.
- A typical input $G$ has other $k$-witnesses than $k$-cliques.
- Need to "dilute" $G$.

We need to encode \textbf{choices} and have a way to \textbf{propagate} them.
Let’s be more specific

**Starting point:** *k*-**Multicolored Subgraph Isomorphism**;
Given a *k*-colored graph \( G = (V_1 \cup \ldots \cup V_k, E) \) and a cubic graph \( H = ([k], F) \), is there a \( i \in [k] \mapsto v_i \in V_i \) s.t. \( ij \in F \Rightarrow v_i v_j \in E \)?
Let’s be more specific

**Starting point:** $k$-**Multicolored Subgraph Isomorphism**;
Given a $k$-colored graph $G = (V_1 \cup \ldots \cup V_k, E)$ and a cubic graph $H = ([k], F)$, is there a $i \in [k] \mapsto v_i \in V_i$ s.t. $ij \in F \Rightarrow v_iv_j \in E$?

**Choice encoding:** An independent set of size $|V_i|$ where
"coloring 1 the $p$-th vertex $\equiv$ mapping $i$ to the $p$-th vertex of $V_i$"

$$V_i^{\text{copy}} = \begin{array}{cccc}
\text{ } & \text{ } & \text{1} & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\end{array} \equiv \begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\end{array} V_i$$
Let’s be more specific

**Starting point:** \textit{k-Multicolored Subgraph Isomorphism}; Given a \(k\)-colored graph \(G = (V_1 \cup \ldots \cup V_k, E)\) and a cubic graph \(H = ([k], F)\), is there a \(i \in [k] \mapsto v_i \in V_i\) s.t. \(ij \in F \Rightarrow v_i v_j \in E\)?

**Choice encoding:** An independent set of size \(|V_i|\) where ”coloring 1 the \(p\)-th vertex \(\equiv\) mapping \(i\) to the \(p\)-th vertex of \(V_i\)”

\[
V_i^{\text{copy}} \quad \begin{array}{c}
\circ \quad 1 \quad \circ \quad \circ \quad \circ \\
\end{array} \quad \equiv \quad \begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \quad \circ \\
i
\end{array} V_i
\]

**Propagation:** Coloring 1 the \(p\)-th vertex in one copy forces it in the next copies

\[
V_i^1 \quad \begin{array}{c}
\circ \quad 1 \quad \circ \quad \circ \quad \circ \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\circ \quad 1 \quad \circ \quad \circ \quad \circ \\
\end{array} V_i^2
\]
Anti-Matching: biclique minus a perfect matching

Would be ideal for propagation
Anti-Matching: biclique minus a perfect matching

But unbounded Grundy number
Biclique

Grundy number 2
Biclique

But do not propagate anything
Simple Half-Graph

A 4 would need to be supported by a 3 on the other side.
Simple Half-Graph

has Grundy number 3
Simple Half-Graph

has Grundy number 3

A 4 would need to be supported by a 3 on the other side
Simple Half-Graph

has Grundy number 3

This would imply on both sides a 2 supported by a 1
Simple Half-Graph

has Grundy number 3

Impossible due to $2K_2$-freeness
Simple Half-Graph

has Grundy number 3

Only half-propagation
Cycles of Half-Graphs

Would yield *full* propagation
Cycles of Half-Graphs

But unbounded Grundy again
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^{\ell}$, proof by induction on $\ell$
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^\ell$, proof by induction on $\ell$

Let $H$ be a minimal colored witness for $\Gamma(A_0 \cup \ldots \cup A_\ell) = \Gamma(G)$
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^\ell$, proof by induction on $\ell$

By hypothesis, $A_0$ and $A_\ell$ contain at least $\Gamma(G) - 4^{\ell-1}$ colors of $H$
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^\ell$, proof by induction on $\ell$

Hence $A_0$ and $A_\ell$ share at least $\Gamma(G) - 2 \cdot 4^{\ell-1} > 4^{\ell-1}$ colors
Length-\( \ell \) Path of Half-Graphs

Grundy number bounded by \( 4^\ell \), proof by induction on \( \ell \)

These color classes restricted to \( G[A_1 \cup \ldots \cup A_\ell] \) form a witness
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^\ell$, proof by induction on $\ell$

Indeed if a color $j$ in $A_1$ is only supported by an $i < j$ in $A_0$
Length-$\ell$ Path of Half-Graphs

Grundy number bounded by $4^\ell$, proof by induction on $\ell$

Then the $j$ appearing in $A_0$ could not be supported by an $i$
Encoding $V_i$ with a length-4 Path of Half-Graphs
Encoding $V_i$ with a length-4 Path of Half-Graphs

1 forced at a pair $l(u_a), r(u_a) \Rightarrow 1$ only at "edges" incident to $u_a
Encoding $V_i$ with a length-4 Path of Half-Graphs

Only useful 1 those with other endpoints selected in their color class
Encoding $V_i$ with a length-4 Path of Half-Graphs

Need gadget which activates iff some pairs are colored 1
Back to binomial trees $T_q = v(T_1, T_2, \ldots, T_{q-1})$

A *unique* optimum Grundy Coloring
Back to binomial trees $T_q = \nu(T_1, T_2, \ldots, T_{q-1})$

Dominant subtree: largest among its siblings
Back to binomial trees $T_q = v(T_1, T_2, \ldots, T_{q-1})$

For each $l(u_a)$, $r(u_a)$, copy tree, remove dominant 1, link to the 2
Back to binomial trees $T_q = v(T_1, T_2, \ldots, T_{q-1})$

Same principle for the edge check with $z(u_a, v_b)$ and $z(v_b, u_a)$
The binomial tree with missing dominant subtrees

Remove $k + 3$ dominant $4$ in the binomial tree $T$ and link each parent $5$ to the roots of a $V_i$ or $E(V_i, V_j)$.
The binomial tree with missing dominant subtrees

$v$ can get color 4 iff its 3 finds a 2 in $S$
The binomial tree with missing dominant subtrees

\[ v \text{ can get color 4 iff its 3 finds a 2 in } S \]

Remove \( k + 3k/2 \) dominant 4 in the binomial tree \( T_{\log k+10} \) and link each parent 5 to the roots of a \( V_i \) or \( E(V_i, V_j) \)
Wrapping up

Deciding if the Grundy number is $\log k + O(1)$ is as hard as \texttt{k-Multicolored Subgraph Isomorphism} for cubic patterns.

\textbf{Theorem (Marx ’10)}

Under the ETH, \texttt{k-Multicolored Subgraph Isomorphism} with cubic patterns cannot be solved in $f(k)n^{o(k/\log k)}$.

Thus,

\textbf{Theorem}

Under the ETH, \texttt{Grundy} cannot be solved in $f(k)n^{2o(k)}$. 
Wrapped up

Deciding if the Grundy number is $\log k + O(1)$ is as hard as $k$-MULTICOLORED SUBGRAPH ISOMORPHISM for cubic patterns.

**Theorem (Marx ’10)**

*Under the ETH, $k$-MULTICOLORED SUBGRAPH ISOMORPHISM with cubic patterns cannot be solved in $f(k)n^{o(k/\log k)}$.***

Thus,

**Theorem**

*Under the ETH, Grundy cannot be solved in $f(k)n^{2^{o(k)}}$.***

Nothing better than trying all colorings of all subsets of size $2^{k-1}$
Summary of our results

- **Grundy Coloring** is \( W[1] \)-hard and unlikely solvable in \( f(k)n^{2^{o(k)}} \)
- **\( b \)-Chromatic Core** is \( W[1] \)-hard:
  - simple half-graphs + Grid Tiling + ad-hoc tricks
- **Partial Grundy Coloring and \( b \)-Chromatic Core** are FPT in \( K_t,t \)-free graphs

**Lemma**
Every \( K_{t,t} \)-free graph with a large number of large-degree vertices admits \( kK_{1,k} \) as an induced sugraph.
Open Questions

- Is Partial Grundy FPT?
- Are Grundy and b-Chromatic Core FPT in $H_{t,t}$-free graphs?
- Is Grundy FPT in $K_{t,t}$-free graphs?
- In general, can some of the FPT algorithms in $K_{t,t}$-free graphs be lifted to $H_{t,t}$-free graphs?
Open Questions

- Is Partial Grundy FPT?
- Are Grundy and b-Chromatic Core FPT in $H_{t,t}$-free graphs?
- Is Grundy FPT in $K_{t,t}$-free graphs?
- In general, can some of the FPT algorithms in $K_{t,t}$-free graphs be lifted to $H_{t,t}$-free graphs?

Thank you for your attention!