The Graph Motif problem parameterized by the structure of its input graph

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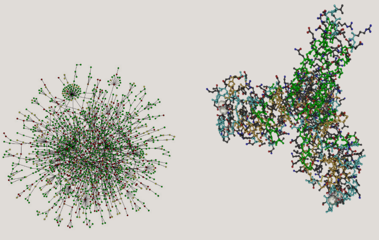
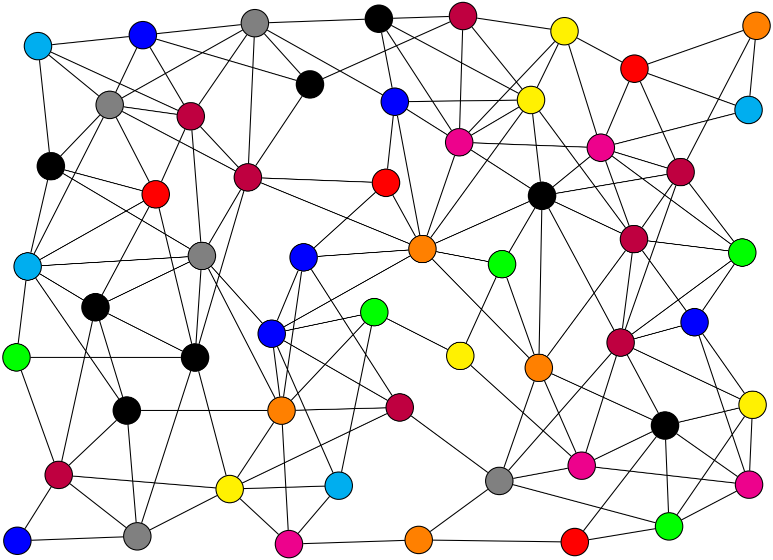
Graph Motif - what is known?

Graph Motif

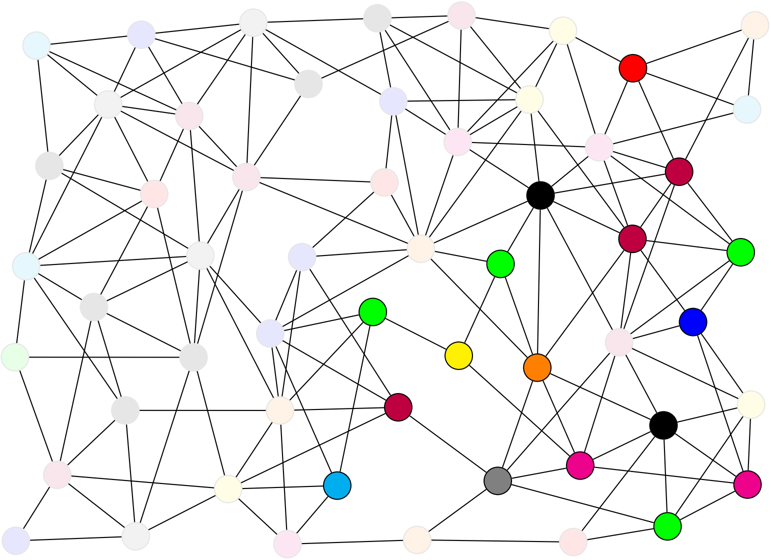


Given a graph $G = (V, E)$ whose vertices are colored by a function $c : V \to \mathcal{C}$ and a multiset M over $\mathcal{C}$ ...



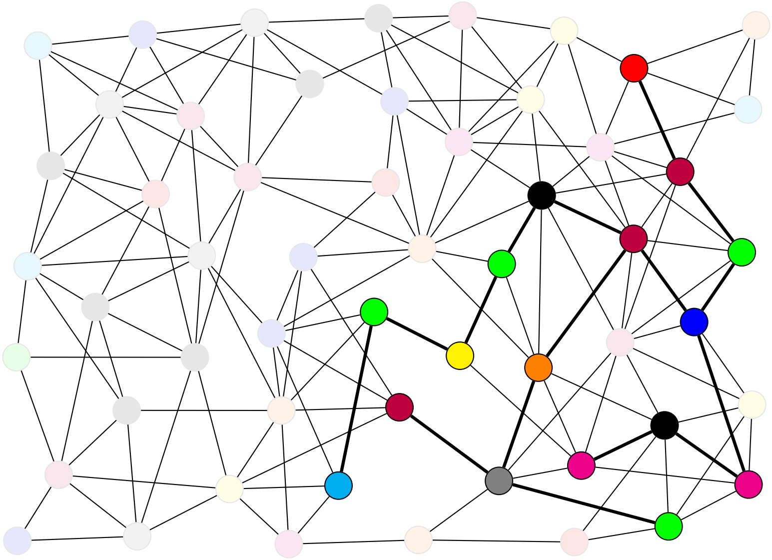
Find a connected subgraph colored by

$M = \{\bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet\}$.



Find a connected subgraph **colored by**

$M = \{$●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●, ●$\}$.



Find a **connected** subgraph colored by

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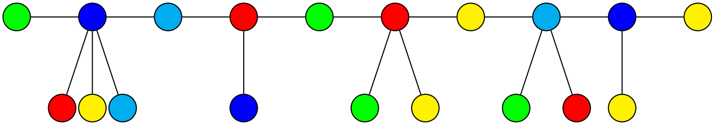
Motivations

- Lacroix et al. (2005): reaction networks.
- Social, technical networks, and mass spectrometry.
- Graph pattern matching with only connectivity constraint.

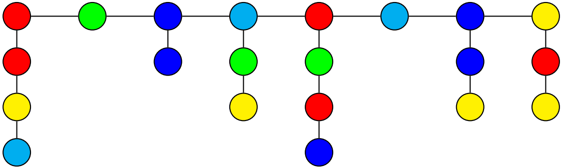
Known results - algorithms

- FPT in the size of the motif: $O^*(2^{|M|})$ [BKK '12,PZ '12].
- FPT in the neighborhood diversity: $O^*(2^{\mathrm{nd}})$ [G '12].
- XP in $\mathrm{tw}(G) + |\mathcal{C}|$ [FFHV '11].
- Polytime solvable in caterpillar trees [ABHKMPR '10].



Known results - hardness

- W[1]-hard on trees w.r.t the number of colors [FFHV '11].
- NP-hard on bipartite graphs of degree 4, $|\mathcal{C}| = 2$ [FFHV '11].
- NP-hard on trees of diameter 4 [ABHKMPR '10].
- NP-hard on comb graphs [CPPW '12].



Neighborhood diversity

- Least number of subsets in a partition into true or false twins.
- κ has linear neighborhood diversity if $\forall G$, $\mathrm{nd}(G) \leqslant O(\kappa(G))$.
- κ has exponential n.d. if $\forall G$, $\mathrm{nd}(G) \leqslant 2^{O(\kappa(G))}$.
- κ has unbounded n.d. if $\forall f$, $\exists G$ such that $\mathrm{nd}(G) > f(\kappa(G))$.

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Vertex cover number has exponential n.d. since $\text{nd}(G) \leqslant \text{vc} + 2^{\text{vc}}$.

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Cluster editing number on connected graphs has linear n.d.: $\text{nd}(G) \leqslant 3k + 1$.

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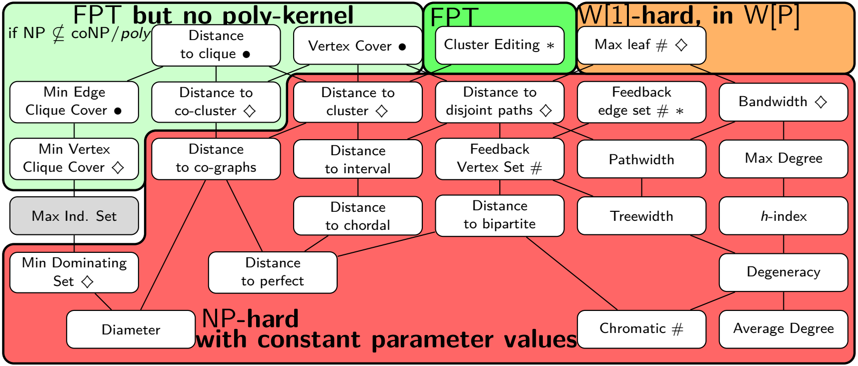
Distance to co-cluster has unbounded n.d.

Aim of the paper

Completing the picture for structural/secondary parameters.

- Does only boundedness of neighborhood diversity count?
- For exponential n.d.: from double to single-exponential time.

Ecological Landscape

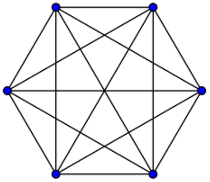


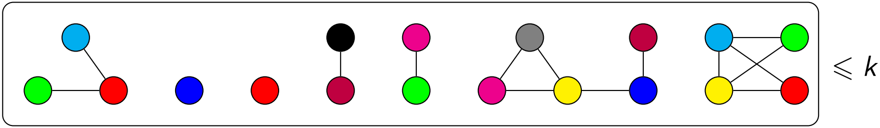
Bounded neighborhood diversity

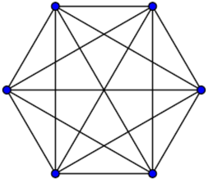
Parameter: distance to clique

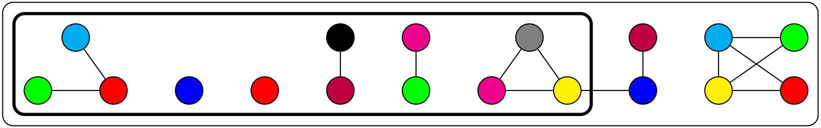
Theorem

GRAPH MOTIF *can be solved in* $O^*(4^k)$.

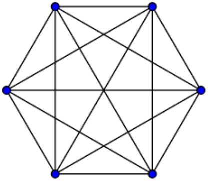


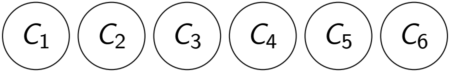

$\leqslant k$





$M' = M -$ 





Set-colored SET COVER with threshold constraints imposed by M'.
Solvable in $O^*(m2^n)$ for m sets and n elements.

Time $O^*(4^k)$ should be improvable to $O^*(2^k)$ but not further:

Observation

Under SCH, Graph Motif *cannot be solved in* $O^*((2-\varepsilon)^k)$*.*

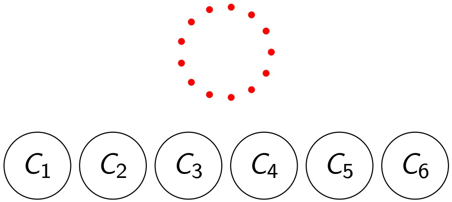
SCH: For any $\varepsilon > 0$, Set Cover cannot be solved in $O((2-\varepsilon)^n)$.

Parameter: vertex cover number

Theorem

GRAPH MOTIF *can be solved in* $O^*(2^{2k \log k})$.

- Let's start as the previous algorithm.



Set-colored SET COVER with threshold constraints **and connected intersection graph of the solution**.

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- Let's start as the previous algorithm.
- Guess an ordered partition of the connected components C_is
- which says how the C_is connect with each other via the IS.

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- Let's start as the previous algorithm.
- Guess an ordered partition of the connected components C_is
- which says how the C_is connect with each other via the IS.
- Maximum matching in an auxiliary bipartite graph.

Again, you may expect to go down to $O^*(2^k)$ but not lower:

Observation

Under SETH, Graph Motif *cannot be solved in* $O^*((2-\varepsilon)^k)$.

SETH: For any $\varepsilon > 0$, SAT cannot be solved in $O((2-\varepsilon)^n)$.

Unbounded neighborhood diversity

Parameter: distance to co-cluster

Theorem

GRAPH MOTIF *can be solved in* $O^*(2^{2k \log k})$.

$H = V - S = I_1 \cup \ldots \cup I_q$ is a co-cluster.
Fix a solution R.

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- case a) R intersects only one I_j, solve $S \cup I_j \rightsquigarrow$ vertex cover k.

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Fix a solution R.

- case a) R intersects only one I_j, solve $S \cup I_j \rightsquigarrow$ vertex cover k.
- case b) R intersects two I_js, cliquify $H \rightsquigarrow$ distance to clique k.

Hardness

Theorem

GRAPH MOTIF *is W[1]-hard for parameter max leaf number.*

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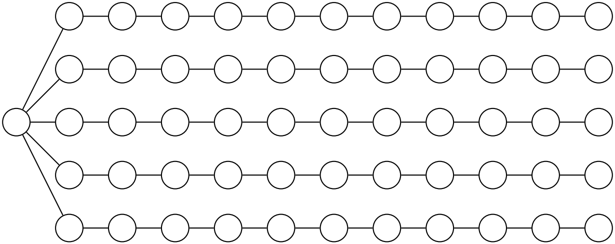
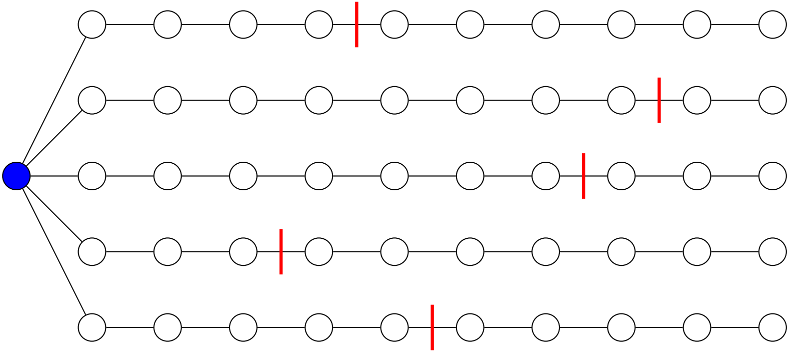
In fact,

Theorem

GRAPH MOTIF *is W[1]-hard for parameter number of leaves of the graph + number of colors in subdivisions of stars.*





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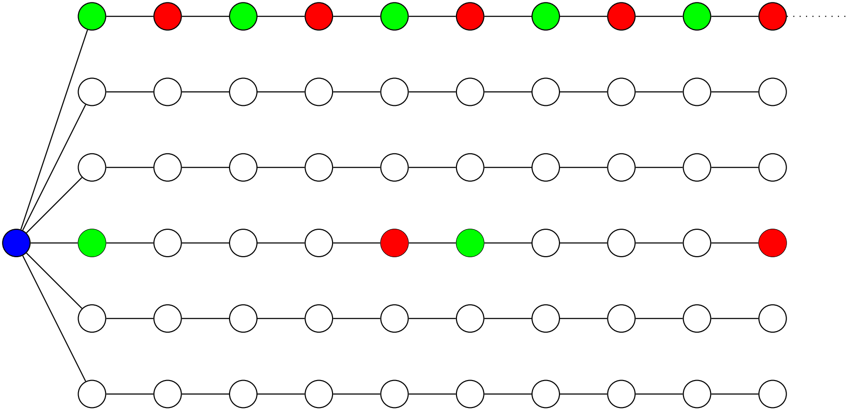
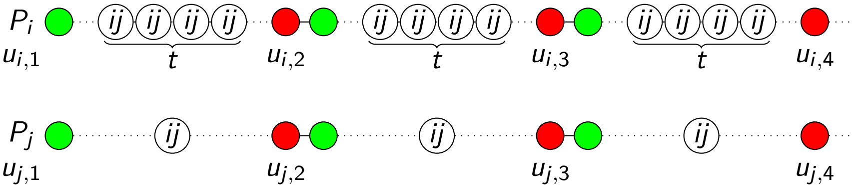


$M = \{$●, ● $\times$ blah, ● $\times$ blah,

Reduction from k-MULTICOLORED CLIQUE on $(\cup_{1 \leqslant i \leqslant k} H_i, E)$ where $H_i = \{u_{i,1}, \ldots, u_{i,t}\}$.



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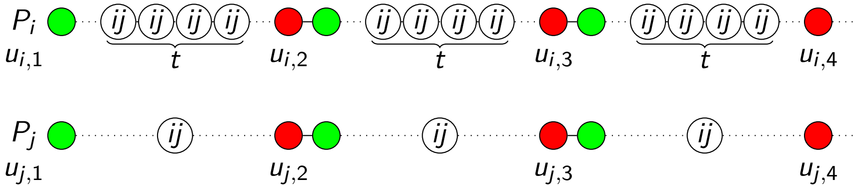

Pairs $(u_{i,a}, u_{j,b})$ and $[t^2 - 1]$ are in one-to-one correspondence.

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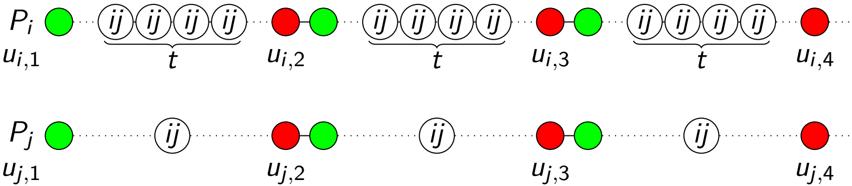

List of edges: $\{13, 15, 20, 29, \ldots, 80, 81, 92, 97\}$ $(t = 10)$.

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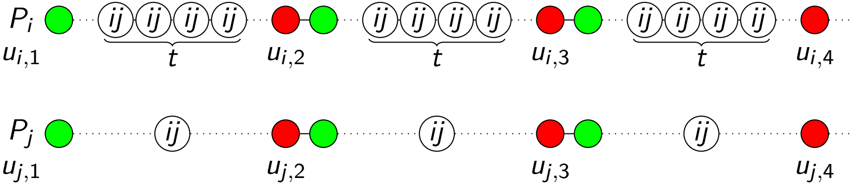

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Complements to t^2: $\{3, 8, 19, 20, \ldots, 71, 80, 85, 87\}$.

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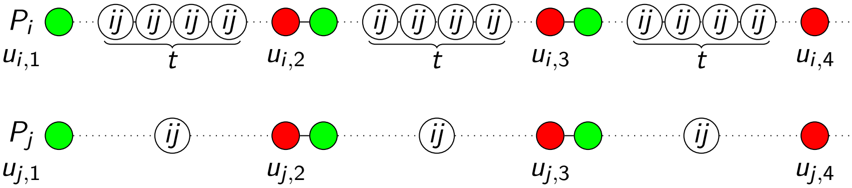

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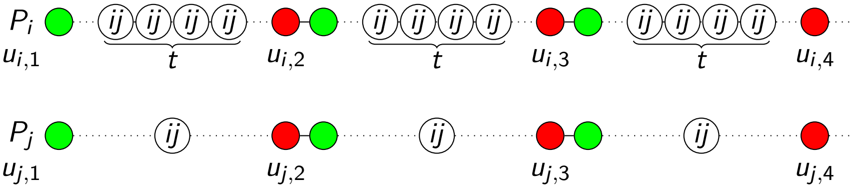

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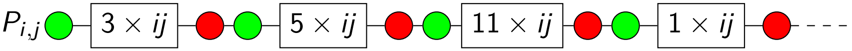

$M = \{\bullet, \bullet \times \text{blah}, \bullet \times \text{blah}, t^2 \times ij\}$

Theorem

Graph Motif *is solvable in* $O^*(16^{ml}n^{10ml}) = n^{O(ml)}$.

Observation

There is less than $4ml$ *vertices of degree at least* 3 *and removing those vertices leaves a disjoint union of at most* $5ml$ *paths.*

The previous reduction was only ruling out $n^{o(\sqrt{\mathrm{ml}})}$ assuming ETH.

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PARTITIONED SUBGRAPH ISOMORPHISM $\rightarrow$ GRAPH MOTIF
$\rightsquigarrow$ $n^{o((\mathrm{ml}(G)+|\mathcal{C}|)/\log(\mathrm{ml}(G)+|\mathcal{C}|))}$ ETH-based lower bound.

Perspectives

- Settle the $2^{O(\mathrm{vc})}$ vs no $2^{o(\mathrm{vc} \log \mathrm{vc})}$ under ETH.
- A $2^{O(\mathrm{vc})}$ algorithm would immediately give a single-exponential for parameter distance to co-cluster and edge clique cover number (where the edge clique cover is given with the input).
- Extend the FPT algorithms to the list-colored variant.