QPTAS and Subexponential Algorithm for Maximum Clique on Disks

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LIP, ENS Lyon

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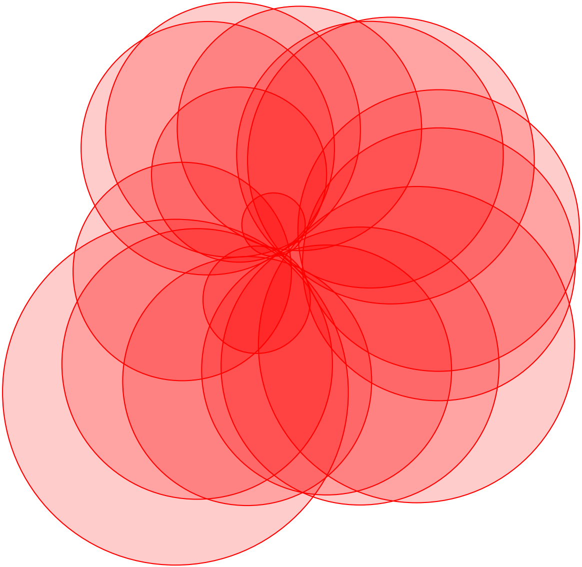
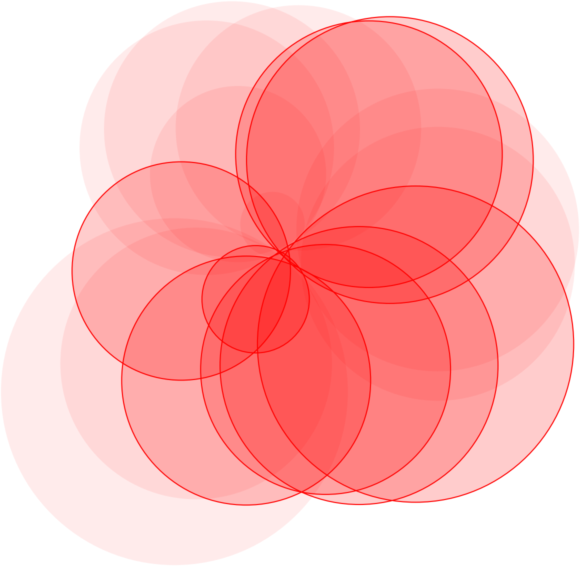
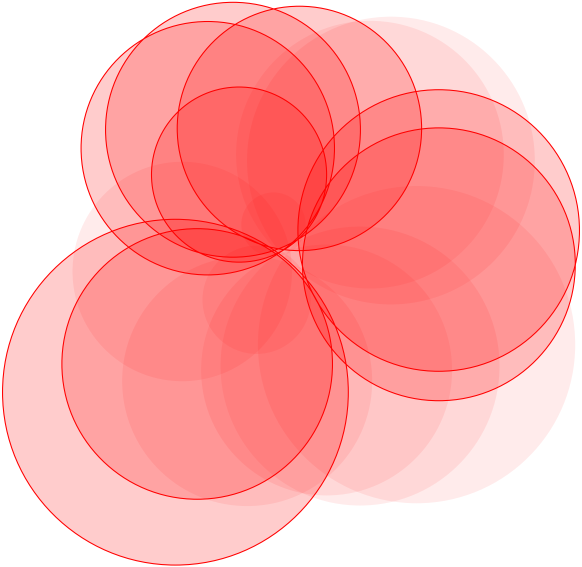


Find a largest collection of disks that pairwise intersect

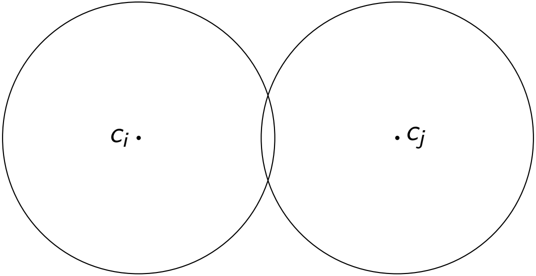


Like this



or that

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



Guess two farthest disks in an optimum solution S.

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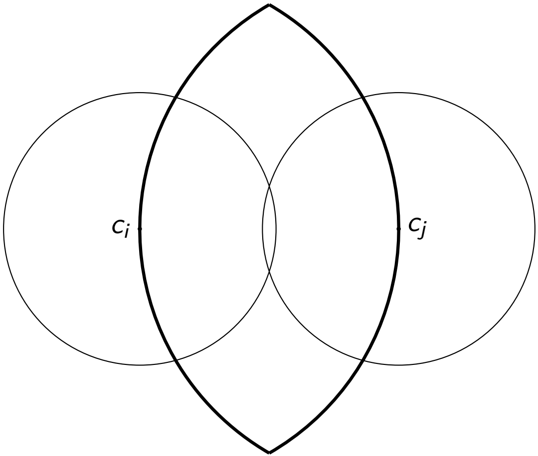

Hence, all the centers of S lie inside the bold digon.

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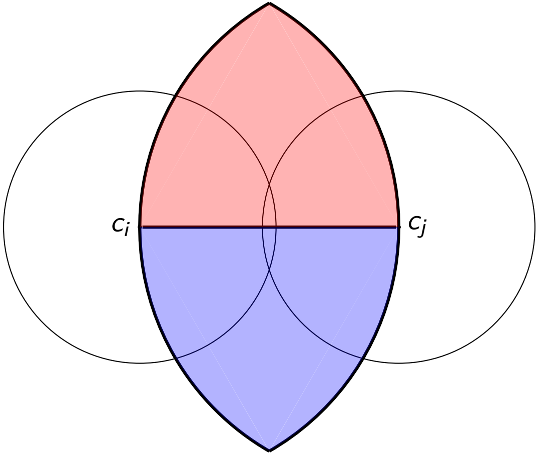

Two disks centered in the same-color region intersect.

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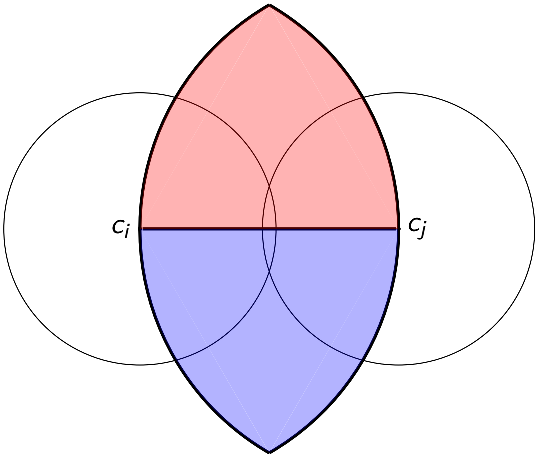

We solve Max Clique in a co-bipartite graph.

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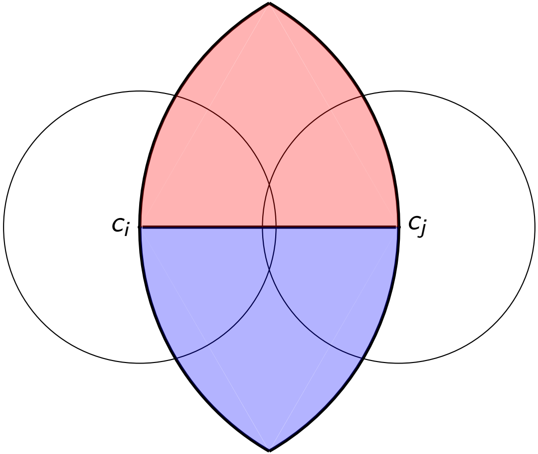


We solve Max Independent Set in a bipartite graph.

Disk graphs

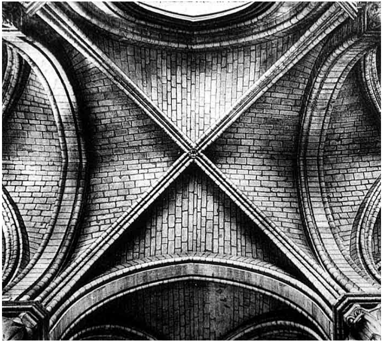
Unweighted problems

3-Colourability [?]	NP-complete	[+]Details
Clique [?]	Unknown to ISGCI	[+]Details
Clique cover [?]	NP-complete	[+]Details
Colourability [?]	NP-complete	[+]Details
Domination [?]	NP-complete	[+]Details
Feedback vertex set [?]	NP-complete	[+]Details
Graph isomorphism [?]	Unknown to ISGCI	[+]Details
Hamiltonian cycle [?]	NP-complete	[+]Details
Hamiltonian path [?]	NP-complete	[+]Details
Independent dominating set [?]	NP-complete	[+]Details
Independent set [?]	NP-complete	[+]Details
Maximum bisection [?]	NP-complete	[+]Details
Maximum cut [?]	NP-complete	[+]Details
Minimum bisection [?]	NP-complete	[+]Details
Monopolarity [?]	NP-complete	[+]Details
Polarity [?]	NP-complete	[+]Details
Recognition [?]	NP-hard	[+]Details

Inherits the NP-hardness of planar graphs.

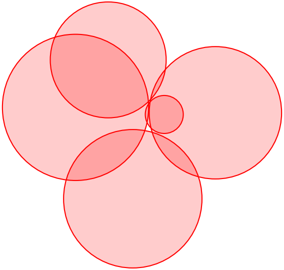
So what is known for Max Clique on disk graphs?

- Polynomial-time 2-approximation
  - For any clique there are 4 points hitting all the disks.
  - Guess those points.
  - Solve exactly in each of the $\binom{4}{2}$ co-bipartite graphs.
  - Output the best solution.
- No non-trivial exact algorithm known.



And what is known about disk graphs?

- Every planar graph is a disk graph.
- Every triangle-free disk graph is planar (centers $\rightarrow$ vertices).
- So a triangle-free non-planar graph like $K_{3,3}$ is not disk.
- A subdivision of a non-planar graph is not a disk graph (more generally not a string graph).
- ...



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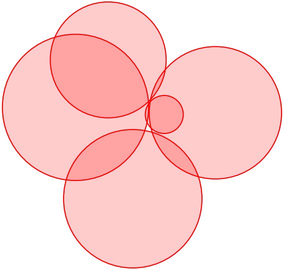
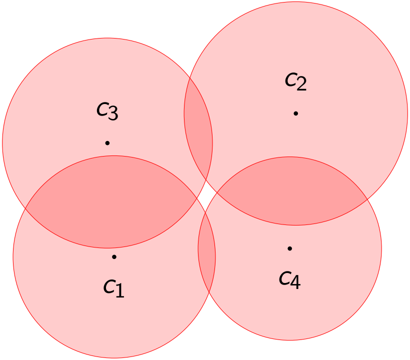


Other ways of showing that a graph is not disk?

Say the 4 centers encoding a $K_{2,2} = \overline{2K_2}$ are in convex position.



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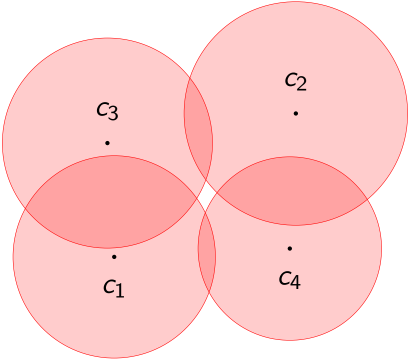

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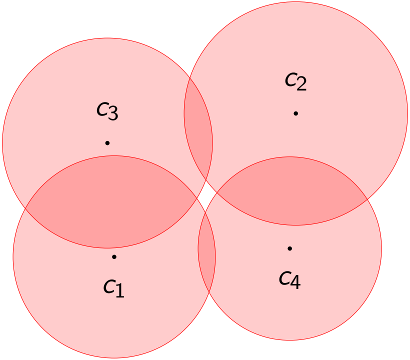
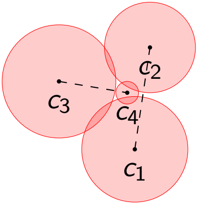


Then the two non-edges should be diagonal.

Suppose $d(c_1, c_3) > r_1 + r_3$ and $d(c_2, c_4) > r_2 + r_4$.
But $d(c_1, c_3) + d(c_2, c_4) \leqslant d(c_1, c_2) + d(c_3, c_4) \leqslant r_1 + r_2 + r_3 + r_4$, a contradiction.

Conclusion: the 4 centers of an induced $\overline{2K_2}$ are either

- not in convex position or
- in convex position with the non-edges being *diagonal*.



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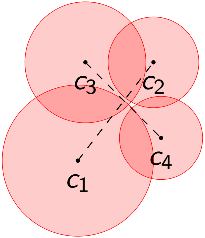

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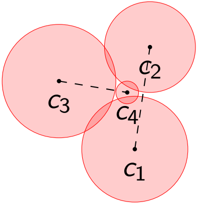

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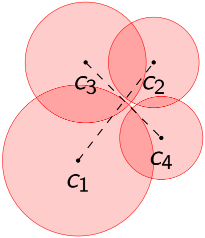
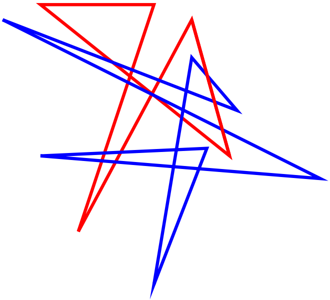


Reformulation: either

- the line $\ell(c_1, c_2)$ crosses the segment c_3c_4, or
- the line $\ell(c_3, c_4)$ crosses the segment c_1c_2, or
- both; equivalently, the segments c_1c_2 and c_3c_4 cross.

Assume $\overline{C_s + C_t}$ is a disk graph.

Link consecutive centers of the two disjoint cycles (non-edges).



For each red segment s_i, we denote by:

- a_i the number of blue segments crossed by $\ell(s_i)$.
- b_i the number of blue segments whose extension cross s_i.
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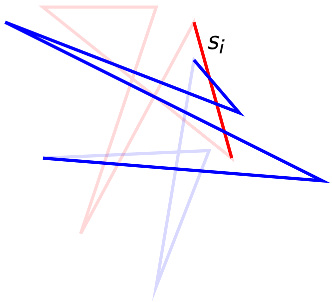

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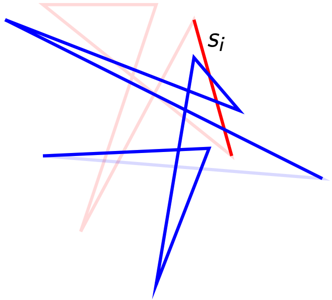

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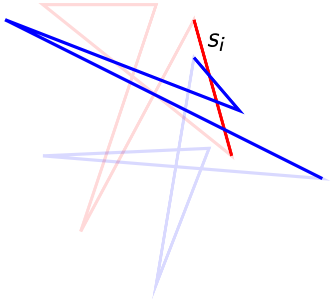

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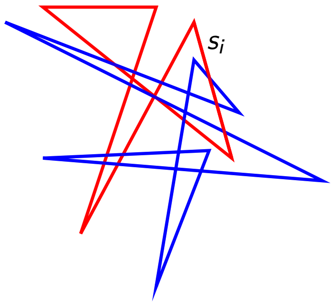

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It should be that $a_i + b_i - c_i = t$.

$$\sum_{1 \leqslant i \leqslant s} a_i + b_i - c_i = st$$

1) a_i is even:

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2) $\sum_{1\leqslant i\leqslant s} b_i = \sum_{1\leqslant i\leqslant t} a'_i$ is therefore even. (a'_j, b'_j, c'_j same for blue segments)

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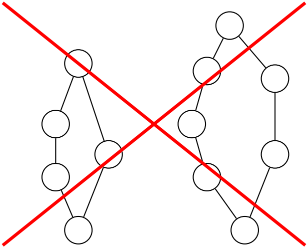
Hence s and t cannot be both odd.

The complement of two odd cycles is not a disk graph.



Going back to algorithms.

Can we solve Max Independent Set more efficiently if there are no two vertex-disjoint odd cycles as an induced subgraph?

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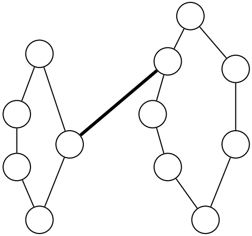
Another way to see it:
at least one edge between two vertex-disjoint odd cycles



Quasi-polynomial time approximation-scheme (QPTAS)

ocp(G): maximum size of an odd cycle packing.

Theorem (Bock et al. 2014)

PTAS for Max Independent Set for $ocp = o(n/\log n)$.

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PTAS for Max Independent Set for ocp $= o(n/\log n)$.

Lemma

Let H complement of a disk graph with n vertices.
If $ocp(H) > n/\log^2 n$, then vertex of degree at least $n/\log^4 n$.

Proof.

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Branching factor $(1, n/\log^4 n)$ (in $2^{\log^5 n}$), and PTAS otherwise.

Subexponential algorithm

Theorem (Györi et al. 1997)

Graphs with odd girth $\geqslant c$ *have odd cycle cover size* $O(\frac{n}{c} \log \frac{n}{c})$.

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$2^{\tilde{O}(\min(n/\Delta, n/c, c\Delta))} \leqslant 2^{\tilde{O}(n^{2/3})}$ for $\Delta = c = n^{1/3}$.

Filled ellipses and triangles

2-subdivisions: graphs where each edge is subdivided exactly twice
co-2-subdivisions: complements of 2-subdivisions

Theorem (technical)

For some α, Maximum Independent Set on 2-subdivisions is not α-approximable algorithm in $2^{n^{1-\varepsilon}}$, unless the ETH fails.

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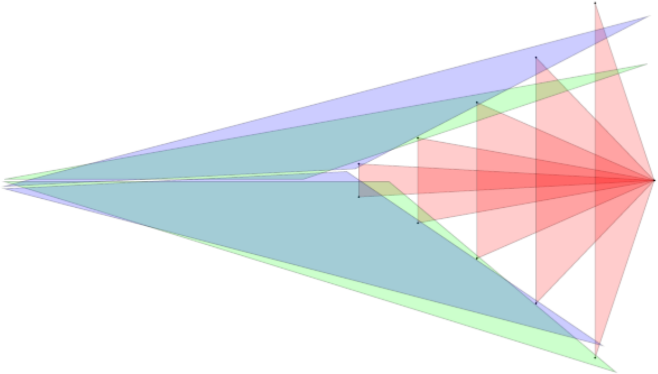
Graphs of filled ellipses or filled triangles contain all the co-2-subdivisions.

Filled triangles



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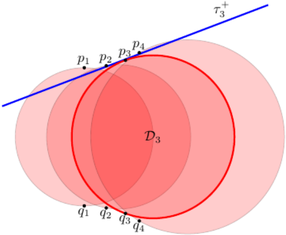

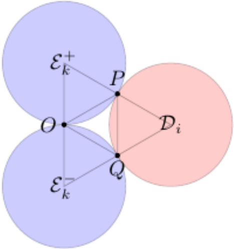

Perspectives

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- Is Max Clique NP-hard on disk graphs?
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Thank you for your attention!