Fine-grained complexity of coloring geometric intersection graphs

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NP-hardness vs ETH-hardness

NP-hardness:

your problem is not solvable in polynomial, unless 3-SAT is very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$. 
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ETH-hardness:

stronger assumption than $P \neq NP$ is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT

Allows to prove stronger conditional lower bounds

linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.
Square root phenomenon on planar graphs

Many problems are solvable in $2^{O(\sqrt{n})}$ in planar graphs, and unlikely solvable in $2^{o(n)}$ in general graphs.
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Dynamic programming would spare a \( \log n \) in the exponent.
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Coloring Unit Disks

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Balanced separators

Theorem (Smith, Wormald '98)

For every $d \geq 1$ and $B \geq 0$, there exists a constant $c = c(d, B)$, such that for every $B$-fat collection $S$ of $n$ $d$-dimensional convex sets with ply at most $\ell$, there exists a $d$-dimensional sphere $Q$, such that:

- at most $\frac{d+1}{d+2} n$ elements of $S$ are entirely inside $Q$,
- at most $\frac{d+1}{d+2} n$ elements of $S$ are entirely outside $Q$,
- at most $cn^{1-1/d} \ell^{1/d}$ elements of $S$ intersect $Q$. 
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- at most \( \frac{d+1}{d+2} n \) elements of \( S \) are entirely outside \( Q \),
- at most \( cn^{1-1/d} \ell^{1/d} \) elements of \( S \) intersect \( Q \).

**ply**: maximum number of objects covering a point.

**\( B \)-fat** objects: aspect ratios diameter/width are bounded by \( B \).
Balanced separators for unit disks

Theorem (Smith, Wormald ’98, special case)

Given a collection $S$ of $n$ disks with ply at most $\ell$, there exists a circle $Q$, such that:

- at most $\frac{3n}{4}$ disks of $S$ are entirely inside $Q$,
- at most $\frac{3n}{4}$ disks of $S$ are entirely outside $Q$,
- at most $O(\sqrt{n\ell})$ disks of $S$ intersect $Q$. 
Standard algorithm for $\ell$-coloring (for unit disks)

If the ply is greater than $\ell$, then more than $\ell$ colors are needed.

Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell}} \log n)$.

Trying all the $\ell$-colorings on $S$ takes time $O(2^{\sqrt{n\ell}} \log \ell)$. 
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Overall running time: $O(2^{\sqrt{n\ell} \log n})$. 
We will see that this running time is optimal up to logarithmic factors in the exponent.
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**Theorem**

*For any $\alpha \in [0, 1]$, coloring $n$ unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.*
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Linear number of colors $\rightsquigarrow$ no subexponential-time algorithm.
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Constant number of colors $\leadsto$ square root phenomenon.
Linear number of colors $\leadsto$ no subexponential-time algorithm.

And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$. 
Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-Grid Coloring

**Input:** An induced subgraph $G$ of the $g \times g$-grid, a positive integer $\ell$. Each cell of this grid is mapped to a set of $\ell$ points (in a smaller grid $[\ell]^2$).

**Question:** Is there an $\ell$-coloring of all the points such that:

- two points in the same cell get different colors;
- if $v$ and $w$ are adjacent in $G$, say, $w = v + (1, 0)$, $p$, resp. $q$, are points in the smaller grid of $v$ resp. $w$, receiving the same color, then $q$ has at a second coordinate which is at least the second coordinate of $p$?
2-Grid 3-SAT

Input: A $g \times g$ grid, a positive integer $k$, each vertex (or cell) of the grid is associated to $k$ variables, and a set $C$ of constraints of two kinds:

- **clause constraints**: for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints**: for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

Question: Is there an assignment of the variables such that all constraints are satisfied?
3-SAT $\rightarrow$ 2-Grid 3-SAT

3-SAT on $N$ variables with bounded number of occurrences (Sparsification Lemma) $\sim$

- split the clauses into $\approx g$ blocks $\sim$
- split again the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2 k$.

$N = \Theta(gk) = \Theta(\sqrt{nk})$
2-Grid 3-SAT → Partial 2-Grid Coloring

- clause checking gadget
- even variable assignment cell
- odd variable assignment cell
- local reference cell
- wires
- consistency checking gadget
Encoding information and reference coloring
Wires

\[ A \]

\[ B \]

\[ \begin{array}{cccc}
  p_1 & p_2 & p_3 & p_4 \\
  q_1 & q_2 & q_3 & q_4 \\
\end{array} \]
Permutation

\[a\quad b\quad c\quad d\]

\[A\]

\[a\quad b\quad c\quad d\]

\[B\]

\[c\quad d\quad a\quad b\]

\[C\]

\[c\quad d\quad a\quad b\]
Forget

\begin{figure}
\centering
\begin{tikzpicture}[scale=0.8]
\draw (0,0) grid (3,3);
\node at (0.5,3.5) {A};
\node at (1.5,3.5) {B};
\node at (2.5,3.5) {C};
\node at (1,0) {$a$};
\node at (1,1) {$b$};
\node at (1,2) {$c$};
\node at (1,3) {$d$};
\node at (1.5,0) {$a|b$};
\node at (1.5,1) {$a|b$};
\node at (1.5,2) {$a|b$};
\node at (1.5,3) {$d$};
\node at (2,0) {$a|b$};
\node at (2,1) {$a|b$};
\node at (2,2) {$c$};
\node at (2,3) {$d$};
\end{tikzpicture}
\end{figure}
Independence
Clauses
Consistency gadget (also crossing)

\[ A(v) \]

\[ A(w) \]

- even variable assignment cell
- odd variable assignment cell
- local reference cell
- combined assignment
- forget
- clause
- wires
- variable assignments
- bottom of reference coloring
- top of reference coloring
Higher dimension

Theorem

For $\alpha \in [0, 1]$ and dimension $d \geq 2$, coloring $n$ unit $d$-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^n \frac{d-1+\alpha}{d}$ for any $\epsilon > 0$, under the ETH.

The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.
(Longer and longer) Segments

Theorem
6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.

Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.
The $x_i$’s lists are $[1, 2, 3]$, the $y_j$’s lists are $[4, 5, 6]$.
Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.
Equality

<table>
<thead>
<tr>
<th>vertex</th>
<th>list</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>$y_i$</td>
<td>4, 5, 6</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1, 4</td>
</tr>
<tr>
<td>$b_1$</td>
<td>4, 5</td>
</tr>
<tr>
<td>$c_1$</td>
<td>4, 6</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2, 5</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4, 5</td>
</tr>
<tr>
<td>$c_2$</td>
<td>5, 6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>3, 6</td>
</tr>
<tr>
<td>$b_3$</td>
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</tr>
<tr>
<td>$y_j$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$x'$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$p_1$</td>
<td>1,5</td>
</tr>
<tr>
<td>$p_2$</td>
<td>1,6</td>
</tr>
<tr>
<td>$q_1$</td>
<td>2,4</td>
</tr>
<tr>
<td>$q_2$</td>
<td>2,6</td>
</tr>
<tr>
<td>$r_1$</td>
<td>3,4</td>
</tr>
<tr>
<td>$r_2$</td>
<td>3,5</td>
</tr>
</tbody>
</table>
Inequality

Some extra gadgets permit to remove the lists.
Same lower bound for 4 colors.
What happens with 3-colors? (whiteboard)
Thanks for your attention!