Fine-grained complexity of coloring geometric intersection graphs

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ToCAI, January 27th
NP-hardness vs ETH-hardness

NP-hardness:

your problem is not solvable in polynomial, unless 3-SAT is very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$. 
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ETH-hardness:

stronger assumption than $P \neq NP$ is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT.

Allows to prove stronger conditional lower bounds
linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.
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Square root phenomenon on planar graphs

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Dynamic programming would spare a $\log n$ in the exponent.
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Frequency assignment in broadcast networks

\( k\)-Coloring is NP-hard for any integer \( k \geq 3 \)

the problem can be 3-approximated
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For general graphs, the answer is yes: for any integer \( k \),
there is an \( O^*(2^n) \) algorithm for \( k\text{-Coloring} \)
and no \( 2^{o(n)} \) algorithm under the ETH.
For planar graphs,
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there is an \( O^*(2^n) \) algorithm for \textit{k-Coloring}

and no \( 2^{o(n)} \) algorithm under the ETH.

For planar graphs, only \textit{3-Coloring} is hard!
Balanced separators for unit disks

Theorem (Smith, Wormald ’98, special case)

Given a collection $S$ of $n$ disks with ply at most $\ell$, there exists a circle $Q$, such that:

- at most $3n/4$ disks of $S$ are entirely inside $Q$,
- at most $3n/4$ disks of $S$ are entirely outside $Q$,
- at most $O(\sqrt{n\ell})$ disks of $S$ intersect $Q$. 
Standard algorithm for $\ell$-coloring (for unit disks)

If the ply is greater than $\ell$, then more than $\ell$ colors are needed. Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell}} \log n)$. Trying all the $\ell$-colorings on $S$ takes time $O(2^{\sqrt{n\ell}} \log \ell)$. 
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Trying all the $\ell$-colorings on $S$ takes time $O(2^{\sqrt{nl}} \log \ell)$.

Overall running time: $O(2^{\sqrt{nl}} \log n)$. 
We will see that this running time is optimal up to logarithmic factors in the exponent.
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**Theorem**

*For any $\alpha \in [0, 1]$, coloring $n$ unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.*
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Linear number of colors $\leadsto$ no subexponential-time algorithm.

And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$. 
Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-Grid Coloring

**Input:** An induced subgraph $G$ of the $g \times g$-grid, a positive integer $\ell$. Each cell of this grid is mapped to a set of $\ell$ points (in a smaller grid $[\ell]^2$).

**Question:** Is there an $\ell$-coloring of all the points such that:
- two points in the same cell get different colors;
- if $v$ and $w$ are adjacent in $G$, say, $w = v + (1, 0)$, $p$, resp. $q$, are points in the smaller grid of $v$ resp. $w$, receiving the same color, then $q$ has at a second coordinate which is at least the second coordinate of $p$?
2-Grid 3-SAT

**Input:** A \( g \times g \) grid, a positive integer \( k \), each vertex (or cell) of the grid is associated to \( k \) variables, and a set \( C \) of constraints of two kinds:

- **clause constraints:** for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints:** for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

**Question:** Is there an assignment of the variables such that all constraints are satisfied?
3-SAT $\rightarrow$ 2-Grid 3-SAT

3-SAT on $N$ variables with bounded number of occurrences (Sparsification Lemma) $\leadsto$

split the variables into $\approx k$ blocks $\leadsto$ split the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2k$.

$N = \Theta(gk) = \Theta(\sqrt{nk})$
2-Grid 3-SAT $\rightarrow$ Partial 2-Grid Coloring

- clause checking gadget
- even variable assignment cell
- odd variable assignment cell
- local reference cell
- wires
- consistency checking gadget
Encoding information and reference coloring
Wires

A

B

p_1
p_2
p_3
p_4

q_1
q_2
q_3
q_4
Permutation

A

B

C

a
b
c
d
a
b
c
d
d
c
a
b
A
B
C
Forget

\[
\begin{array}{cc}
A & B \\
\begin{array}{c}
\bullet a \\
\bullet b \\
\bullet c \\
\bullet d \\
\end{array} & \begin{array}{c}
\bullet a | b \\
\bullet a | b \\
\bullet c \\
\bullet d \\
\end{array}
\end{array}
\]

\[
\begin{array}{cc}
C & \\
\begin{array}{c}
\bullet a | b \\
\bullet a | b \\
\bullet c \\
\bullet d \\
\end{array}
\end{array}
\]
Independence
Clauses

reference coloring

R

S

T

U

B

A

x1

x2

x3

variable assignment

clause gadget

variable assignment

[6] \ c
Consistency gadget (also crossing)
Higher dimension

Theorem
For $\alpha \in [0, 1]$ and dimension $d \geq 2$, coloring $n$ unit $d$-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{n^{d-1+\alpha/d - \epsilon}}$ for any $\epsilon > 0$, under the ETH.

The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.
(Longer and longer) Segments

**Theorem**

6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.

Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.

The $x_i$'s lists are [1, 2, 3], the $y_j$'s lists are [4, 5, 6].

Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.
Equality

\begin{align*}
\text{vertex} & \quad \text{list} \\
(x_i) & \quad 1,2,3 \\
(y_i) & \quad 4,5,6 \\
a_1 & \quad 1,4 \\
b_1 & \quad 4,5 \\
c_1 & \quad 4,6 \\
a_2 & \quad 2,5 \\
b_2 & \quad 4,5 \\
c_2 & \quad 5,6 \\
a_3 & \quad 3,6 \\
b_3 & \quad 4,6 \\
c_3 & \quad 5,6 
\end{align*}
Inequality

<table>
<thead>
<tr>
<th>vertex</th>
<th>list</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>1,2,3</td>
</tr>
<tr>
<td>$y_j$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$x'$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$p_1$</td>
<td>1,5</td>
</tr>
<tr>
<td>$p_2$</td>
<td>1,6</td>
</tr>
<tr>
<td>$q_1$</td>
<td>2,4</td>
</tr>
<tr>
<td>$q_2$</td>
<td>2,6</td>
</tr>
<tr>
<td>$r_1$</td>
<td>3,4</td>
</tr>
<tr>
<td>$r_2$</td>
<td>3,5</td>
</tr>
</tbody>
</table>
Inequality

Some extra gadgets permit to remove the lists.
Same lower bound for 4 colors.
What happens with 3-colors? (whiteboard)
Thanks for your attention!