Polyspace slightly superexponential parameterized algorithm for Subgraph Isomorphism in proper-minor closed classes

Algorithmic application in Pilipczuk and Siebertz’s paper on $p$-centered coloring

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Subgraph Isomorphism

Is $H$ a subgraph of $G$?
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Complexity of \textbf{Subgraph Isomorphism}

“Is $H$ in $G$?” generalizes $k$-$\text{Clique}$

NP-complete, $W[1]$-complete parameterized by $p := |V(H)|$
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no \( f(p)n^{o(p)} \), no \( f(p)n^{o(p/\log p)} \) for cubic \( H \), with \( n := |V(G)| \)
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\( 2^{O(\sqrt{p}\log^2 p)}n^{O(1)} \), if further \( H \) has bounded-degree
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In sharp contrast: no $2^{o(n/\log n)}$, when $G$ is series-parallel, $H$ is a tree, and both graphs have only one vertex of degree more than 3.
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**Theorem (Pilipczuk, Siebertz ’19)**

Subgraph Isomorphism *can be solved in time* $2^{O(p \log p)}n^{O(1)}$
*and polynomial space*, when $G$ is $K_t$-minor free.
Treedept

**Treedept of** $G$: smallest height of a forest $F$ such that $G$ is a subgraph of the ancestor-descendant closure of $F$.

![Diagram of forest $F$]
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![Diagram](clos(F))
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\[ F \text{ clos}(F) \]

\( G \) has treedefth 4
\textbf{$p$-Centered colorings are treedepth-$p$ colorings}

\textbf{Treedepth-$p$ coloring:} every subgraph induced by a set $X$ of at most $p$ colors have treedepth at most $|X|$. 
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In each CC of $G$, one color appears exactly once
Set the corresponding vertices to be roots of the forest
\textbf{p-Centered colorings are treedepth-}p\ \textbf{colorings}

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In\ each\ CC\ of\ \(G\),\ one\ color\ appears\ exactly\ once
Set\ the\ corresponding\ vertices\ to\ be\ roots\ of\ the\ forest.

The\ rest\ of\ the\ CC\ has\ at\ most\ \(|X|−1\)\ colors → recurse.
Reducing to graphs of treedepth $\leq p := |V(H)|$

$G \in \mathcal{C}$ excluding a minor $\rightarrow n^{O(1)}$ $p$-centered coloring with $p^{O(1)}$ colors
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$\forall$ color set $X$ of size $p$: treedepth decomposition $F$ of $G' := G[\{v|v \text{ has a color in } X\}]$
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$\rightarrow (p^{O(1) \choose p})^{n^{O(1)}} = 2^{O(p \log p)} n^{O(1)}$
Reducing to graphs of treedepth $\leq p := |V(H)|$

$G \in \mathcal{C}$ excluding a minor $\rightarrow$ $p$-centered coloring with $p^{O(1)}$ colors

$\forall$ color set $X$ of size $p$:

- treedepth decomposition $F$ of $G' := G[\{v | v \text{ has a color in } X\}]$
- $\rightarrow \left(\frac{p^{O(1)}}{p}\right)^{n^{O(1)}} = 2^{O(p \log p)} n^{O(1)}$

Solve "is $H$ in $G'$?" helped by $F$
Reducing to graphs of treedepth \( \leq p := |V(H)| \)

\[ G \in \mathcal{C} \text{ excluding a minor } \rightarrow p\text{-centered coloring with } p^{O(1)} \text{ colors} \]

\[ \forall \text{ color set } X \text{ of size } p: \]
\[ \text{treedepth decomposition } F \text{ of } G' := G[\{ v \mid v \text{ has a color in } X \}] \]
\[ \rightarrow (p^{O(1)}) n^{O(1)} = 2^{O(p \log p)} n^{O(1)} \]

**Solve “is } H \text{ in } G'?” helped by } F \]

A solution cannot escape since it receives at most } p \text{ colors}
Color coding step

\[ H \]

\[ G \]
Color coding step

Give each vertex a random color between 1 and $p$
Color coding step

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Repeating this $100p^p$ times, well color a solution with prob. 0.999
Color coding step

Repeating this $p^p n$ times, well color a solution a.a.s.
Theorem (Alon, Yuster, Zwick ’95)

One can compute in polynomial-time a family \( \mathcal{F} \) of \( p^{O(1)} \log n \) functions \( f : V(G) \to \{1, \ldots, p^2\} \) such that for every set \( X \subseteq V(G) \) of size \( p \) there exists \( f \in \mathcal{F} \) injective on \( X \).
Derandomization

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$p$-perfect family: every vertex subset of size $p$ is multicolored (no repetition of colors) by at least one function of the family.
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$p$-perfect family: every vertex subset of size $p$ is multicolored (no repetition of colors) by at least one function of the family.

Theorem (Schmidt, Siegal ’90)

One can compute in polynomial-time a family $\mathcal{G}$ of $2^{O(p)}$ functions $f : \{1, \ldots, p^2\} \to \{1, \ldots, p\}$ such that for every set $X \subseteq \{1, \ldots, p^2\}$ of size $p$ there exists $g \in \mathcal{G}$ injective on $X$.

$\mathcal{F}' = \{\sigma \circ g \circ f | f \in \mathcal{F} \text{ and } g \in \mathcal{G} \text{ and } \sigma \in S_p\}$

$|\mathcal{F}'| = p! \cdot 2^{O(p)} \cdot p^{O(1) \log n} = 2^{O(p \log p) \log n}$
Colored Subgraph Isomorphism on bounded treedepth graphs

We are now left with proving:

Theorem (Pilipczuk, Siebertz ’19)

Colored Subgraph Isomorphism can be solved in time $2^{O(p \log p)} n^{O(1)}$ and polynomial space, when $G$ is given with a treedepth decomposition of depth at most $p$. 

\begin{figure}
\centering
\begin{tikzpicture}
  \node [draw,circle,fill=red,minimum size=1cm] (a) at (0,0) {$H$};
  \node [draw,circle,fill=green,minimum size=1cm] (b) at (1,0) {};
  \node [draw,circle,fill=blue,minimum size=1cm] (c) at (2,0) {};
  \node [draw,circle,fill=purple,minimum size=1cm] (d) at (1,1) {};

  \node [draw,circle,fill=red,minimum size=1cm] (e) at (4,0) {$G$};
  \node [draw,circle,fill=green,minimum size=1cm] (f) at (5,0) {};
  \node [draw,circle,fill=blue,minimum size=1cm] (g) at (6,0) {};
  \node [draw,circle,fill=purple,minimum size=1cm] (h) at (5,1) {};
  \node [draw,circle,fill=red,minimum size=1cm] (i) at (6,1) {};
  \node [draw,circle,fill=yellow,minimum size=1cm] (j) at (5,2) {};

  \draw (a) -- (b) -- (c) -- (d) -- (a);
  \draw (a) -- (e) -- (g) -- (i) -- (j) -- (f);
  \draw (b) -- (d);
  \draw (c) -- (e) -- (f);
  \draw (d) -- (j);
  \draw (e) -- (h) -- (i);
  \draw (f) -- (h);
  \draw (g) -- (i);
  \draw (h) -- (j);

  \node [draw,thick,fill=gray,fill opacity=0.2] (f) at (7,0) {$F$};

  \draw [->] (7.5,0) -- (8.5,0) node [right] {$\leq p$};
\end{tikzpicture}
\end{figure}
Some notations for the upcoming dynamic-programming
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\[
\text{Chld}(u): \text{set of children of } u
\]
Some notations for the upcoming dynamic-programming

$\text{Child}(u)$: set of children of $u$
Some notations for the upcoming
dynamic-programming

Tail($u$): set of strict ancestors of $u$
Some notations for the upcoming dynamic-programming

Desc($u$): set of descendants of $u$, including $u$
disjoint pair \((X, D)\),
\(H[X]\) connected, and
\(N_H(X) \subseteq D\)
A tuple \((u, X, D, \gamma)\) with \(u \in V(G)\) is a chunk \(\gamma: D \to \text{Tail}(u)\) injective.
A tuple \((u, X, D, \gamma)\)

\(u \in V(G)\)

\((X, D)\) is a chunk

\(\gamma: D \to \text{Tail}(u)\) injective
Subproblems

Val\( (u, X, D, \gamma) = \)
Is there \( \gamma' : X \to \text{Desc}(u) \) such that
\( \gamma \cup \gamma' \) is a (color-preserving) subgraph embedding?
Subproblems

Val($u, X, D, \gamma$) =
Is there $\gamma' : X \rightarrow \text{Desc}(u)$ such that $\gamma \cup \gamma'$ is a (color-preserving) subgraph embedding?
How many tuples \((u, X, D, \gamma)\)?
\[
\leq n \cdot 3^p \cdot p^p = 2^{O(p \log p)} n
\]
Computing Val, $u$ is a leaf of $F$

$\text{Val}(u, \emptyset, D, \gamma) = [\gamma \text{ is a subgraph embedding}]$
Computing $\text{Val}$, $u$ is a leaf of $F$

$\text{Val}(u, \emptyset, D, \gamma) = \left[ \gamma \text{ is a subgraph embedding} \right]$  

$\text{Val}(u, \{w\}, D, \gamma) = \left[ u \text{ is colored } w \text{ and } \gamma \cup \{w \rightarrow u\} \text{ is a s.e.} \right]$
Computing Val, $u$ is a not a leaf of $F$

If $u$ has not a color of $X$:

$$\text{Val}(u, X, D, \gamma) = \bigvee_{v \in \text{Child}(u)} \text{Val}(v, X, D, \gamma)$$
Computing $\text{Val}$, $u$ is a *not* a leaf of $F$

If $u$ has *not* a color of $X$:

$$\text{Val}(u, X, D, \gamma) = \bigvee_{v \in \text{Chld}(u)} \text{Val}(v, X, D, \gamma)$$

If $u$ is colored $w \in X$:

$$\text{Val}(u, X, D, \gamma) = \bigvee_{v \in \text{Chld}(u)} \text{Val}(v, X, D, \gamma) \lor$$

$$\bigwedge_{Z \in \text{CC}(X \setminus \{w\})} \bigvee_{v \in \text{Chld}(u)} \text{Val}(v, Z, D \cup \{w\}, \gamma \cup \{w \rightarrow u\})$$
Algorithm

Compute $\bigvee_{r \text{ root of } F} \text{Val}(r, X, \emptyset, \emptyset)$ for every $X$ CC of $H$
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If all positive answers $\rightarrow$ overall solution. Why?
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If all positive answers \( \rightarrow \) overall solution. Why?

**Disjointness.** That was the point of color coding.
Complexity

**Space:** polynomial, calling stack bounded by treedepth
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**Time:** $2^{O(p \log p)} n^{O(1)}$ all recursive calls are different. A non-root call defines a unique parent tuple.
Summary

$p$ color classes of a $p$-centered coloring have treedepth $p$

Color coding for solution disjointness

Treedepth DP allows polyspace, as opposed to treewidth DP

An example of such an algorithm, notion of chunk
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An example of such an algorithm, notion of chunk

Thank you for your attention!