Mixed minors, compact representations and χ-boundedness

Marek Sokołowski

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Part 1

Mixed minors

Twin-width and matrices

A 0/1-matrix can be:

Twin-width and matrices

A 0/1-matrix can be:

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

horizontal

Twin-width and matrices

A 0/1-matrix can be:

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}$$

horizontal vertical

Twin-width and matrices

A 0/1-matrix can be:

0	0	0	0
1	1	1	1
1	1	1	1
0	0	0	0
1	1	1	1

horizontal

0	1	1	0
0	1	1	0
0	1	1	0
0	1	1	0
0	1	1	0

vertical

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

constant

Twin-width and matrices

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$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

horizontal   vertical   constant   mixed

Twin-width and matrices

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$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$
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horizontal | vertical | constant | mixed

Note: mixed $\iff$ has a 2×2 contiguous mixed submatrix (**corner**).

Twin-width and matrices

Divisions

Division $\mathcal{D}$ – partitioning of columns and rows into intervals (*blocks*).

$$
\begin{array}{cc|ccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Twin-width and matrices

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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Zone – intersection of a row block and a column block

Twin-width and matrices

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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Zone – intersection of a row block and a column block

Mixed minors

$\mathcal{D}$ is a **mixed minor** if each zone of $\mathcal{D}$ is mixed.

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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Zone – intersection of a row block and a column block

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$\mathcal{D}$ is a **mixed minor** if each zone of $\mathcal{D}$ is mixed.

Mixed freeness

Matrix M is ***d*-mixed free** if it has no $d \times d$ mixed minor.

Grid theorem for twin-width

Mixed freeness

Matrix M is d-**mixed free** if it has no $d \times d$ mixed minor.

Theorem (Twin-width I)

Let $d \in \mathbb{N}$ be an integer and G be a graph. Then:

- $\operatorname{tww}(G) \leq d \implies G$ has a $(2d+2)$-**mixed free** adjacency matrix.
- G has a d-**mixed free** adjacency matrix $\implies \operatorname{tww}(G) \leq 2^{2^{O(d)}}$.

Marcus–Tardos theorem and twin-width

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Theorem (Twin-width I, "Marcus–Tardos")

If: M – a d-mixed free matrix,
$\mathcal{D}$ – an $n \times n$ division of M

$\Rightarrow \mathcal{D}$ has at most $c_d \cdot n$ mixed zones ($c_d = \mathrm{const}(d)$).

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Number of mixed zones: **linear** instead of **quadratic**!

Part 2

Compact representations

Pilipczuk, Sokołowski, Zych-Pawlewicz,
Compact Representation for Matrices of Bounded Twin-Width

Twin-width of matrices

Twin-width of matrices

t-twin-ordered matrices

0	1	1	0	0	0
1	0	1	1	1	0
0	0	1	0	0	0
0	0	1	0	0	1
1	1	1	1	1	1
0	0	0	1	1	1

Maximum $\times$'s in any row/column now: 0

Maximum so far: 0

Twin-width of matrices

t-twin-ordered matrices

0	1	1	0	0	0
1	0	1	1	1	0
0	0	1	0	0	0
0	0	1	0	0	1
1	1	1	1	1	1
0	0	0	1	1	1

Maximum $\times$'s in any row/column now: 0

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0	1	1	0	0
1	0	1	1	0
0	0	1	0	0
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Maximum $\times$'s in any row/column now: 0

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0	1	1	0	0
1	0	1	1	0
0	0	1	0	0
0	0	1	0	1
1	1	1	1	1
0	0	0	1	1

Maximum $\times$'s in any row/column now: 0

Maximum so far: 0

Twin-width of matrices

t-twin-ordered matrices

0	1	1	0	0
1	0	1	1	0
0	0	1	0	$\times$
1	1	1	1	1
0	0	0	1	1

Maximum $\times$'s in any row/column now: 1

Maximum so far: 1

Twin-width of matrices

t-twin-ordered matrices

0	1	1	0	0
1	0	1	1	0
0	0	1	0	$\times$
1	1	1	1	1
0	0	0	1	1

Maximum $\times$'s in any row/column now: 1
Maximum so far: 1

Twin-width of matrices

t-twin-ordered matrices

0	1	1	0	0
1	0	1	1	0
0	0	1	0	×
1	1	1	1	1
0	0	0	1	1

Maximum ×'s in any row/column now: 1

Maximum so far: 1

Twin-width of matrices

t-twin-ordered matrices

×	1	0	0
×	1	1	0
0	1	0	×
1	1	1	1
0	0	1	1

Maximum ×'s in any row/column now: 2

Maximum so far: 2

Twin-width of matrices

t-twin-ordered matrices

$\times$	1	0	0
$\times$	1	1	0
0	1	0	$\times$
1	1	1	1
0	0	1	1

Maximum $\times$'s in any row/column now: 2
Maximum so far: 2

Twin-width of matrices

t-twin-ordered matrices

×	1	×	0
0	1	0	×
1	1	1	1
0	0	1	1

Maximum ×'s in any row/column now: 2
Maximum so far: 2

Twin-width of matrices

t-twin-ordered matrices

×	1	×	0
0	1	0	×
1	1	1	1
0	0	1	1

Maximum ×'s in any row/column now: 2
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Twin-width of matrices

t-twin-ordered matrices

$\times$	$\times$	0
$\times$	0	$\times$
1	1	1
0	1	1

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Twin-width of matrices

t-twin-ordered matrices

$\times$	$\times$	0
$\times$	0	$\times$
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×	×
1	1
0	1

Maximum ×'s in any row/column now: 2

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Twin-width of matrices

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×	×
1	1
0	1

Maximum ×'s in any row/column now: 2
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Twin-width of matrices

t-twin-ordered matrices

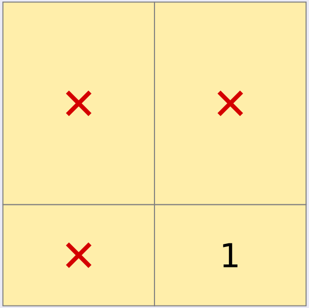
×	×
×	1

Maximum ×'s in any row/column now: 2

Maximum so far: 2

Twin-width of matrices

t-twin-ordered matrices



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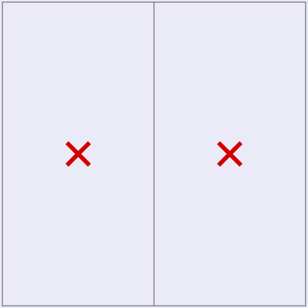
Twin-width of matrices

t-twin-ordered matrices



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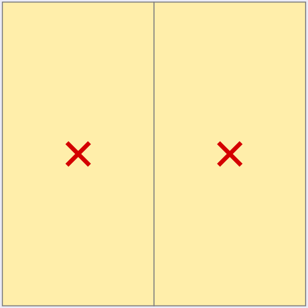

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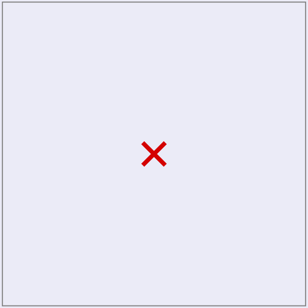

Maximum $\times$'s in any row/column now: 1

Maximum so far: 2

Twin-width of matrices

Red number of a contraction sequence

Red number = maximum $\times$'s in any row/column during the contraction.

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M is d-**twin-ordered** $\iff$ M has a contraction with red number $\leq d$.

Twin-width of matrices

Red number of a contraction sequence

Red number = maximum $\times$'s in any row/column during the contraction.

Twin-width of a matrix

M is **d-twin-ordered** $\iff M$ has a contraction with red number $\leq d$.

$\operatorname{tww}(M) \leq d \iff M$ is **d-twin-ordered** for some permutation of rows/columns.

Twin-width of matrices

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Note: M is d-**twin-ordered** $\implies$ M is $(2d+2)$-**mixed-free**.

Compact data structures

Objectives

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Our case:

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Compact: bitsize $\mathcal{O}(S)$ bits if S = information-theoretic min bitsize.

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$\approx$ Twin-width II

The number of binary d-**twin-ordered** $n \times n$ matrices is $2^{\Theta_d(n)}$.

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	Bitsize	**Query time**
just store the matrix	$\mathcal{O}(n^2)$	$\mathcal{O}(1)$

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	Bitsize	**Query time**
just store the matrix	$\mathcal{O}(n^2)$	$\mathcal{O}(1)$
store the idx of d-twin-ordered matrix	$\mathcal{O}_d(n)$	*huge*

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store the idx of d-twin-ordered matrix	$\mathcal{O}_d(n)$	*huge*
adjacency labeling (Twin-width II)	$\mathcal{O}_d(n \log n)$	$\mathcal{O}_d(\log n)$

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Orthogonal Point Location (Chan, 2013)	$\mathcal{O}_d(n \log n)$	$\mathcal{O}_d(\log \log n)$

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our result (PSZ-P, 2022)	$\mathcal{O}_d(n)$	$\mathcal{O}_d(\log \log n)$

Different zones in a division

M — a d-twin-ordered $n \times n$ matrix;

$s \mid n$;

$\mathcal{D}$ — an $\frac{n}{s} \times \frac{n}{s}$ division of M where each zone is an $s \times s$ submatrix.

1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0
1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1

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M — a d-twin-ordered $n \times n$ matrix;

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Small s

$\mathcal{D}$ has at most $2^{\mathcal{O}_d(s)}$ different zones *(Twin-width II)*.

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$\Longrightarrow$ for $s \ll \log n$, at most $\sqrt{n}$ different matrices of size s.

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We prove: $\mathcal{D}$ has at most $\mathcal{O}_d(\frac{n}{s})$ different zones.

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- "Marcus–Tardos": at most $\mathcal{O}_d(\frac{n}{s})$ **mixed** zones in total;

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We prove: $\mathcal{D}$ has at most $\mathcal{O}_d(\frac{n}{s})$ different zones.

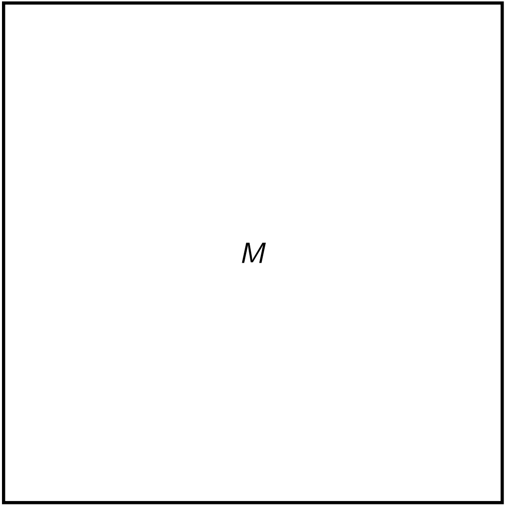
- "Marcus–Tardos": at most $\mathcal{O}_d(\frac{n}{s})$ **mixed** zones in total;
- Now (blackboard): at most $\mathcal{O}_d(\frac{n}{s})$ different **non-mixed** zones.

Data structure

Reminder

- Fixed: $d \in \mathbb{N}$.
- Input: M – an $n \times n$ matrix that is **d-twin-ordered**.
- Target:

  $\mathcal{O}_d(n)$ bits of memory,
  $\mathcal{O}(\log \log n)$ per query.


M

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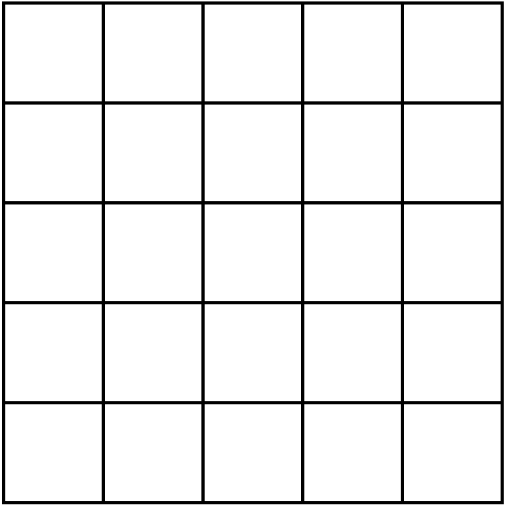

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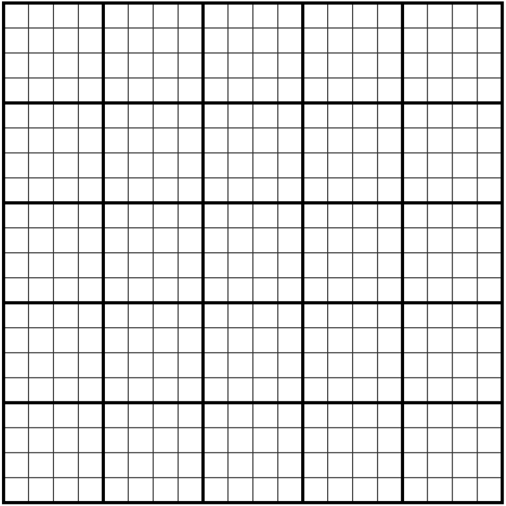

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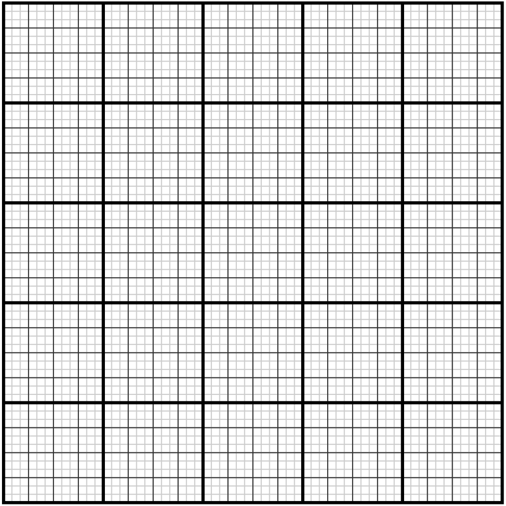

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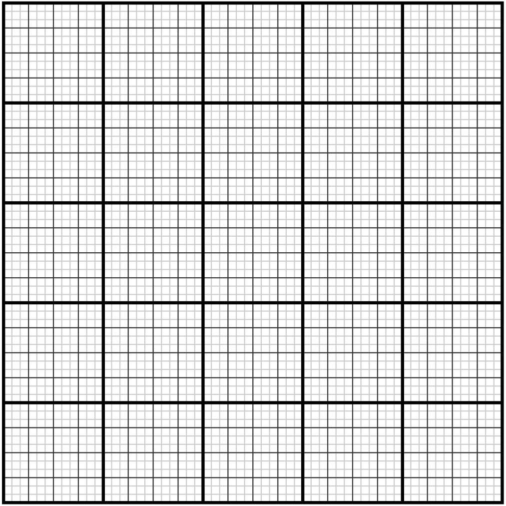

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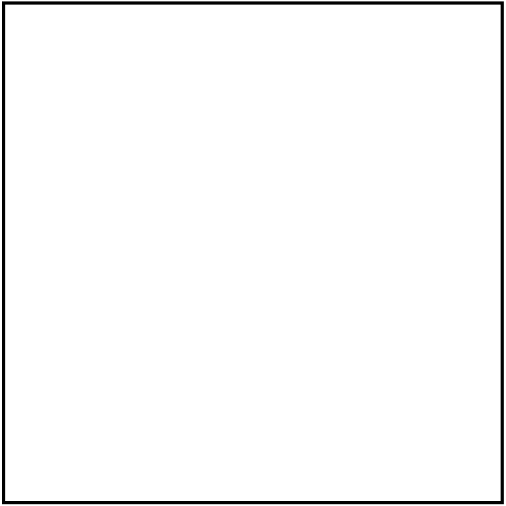

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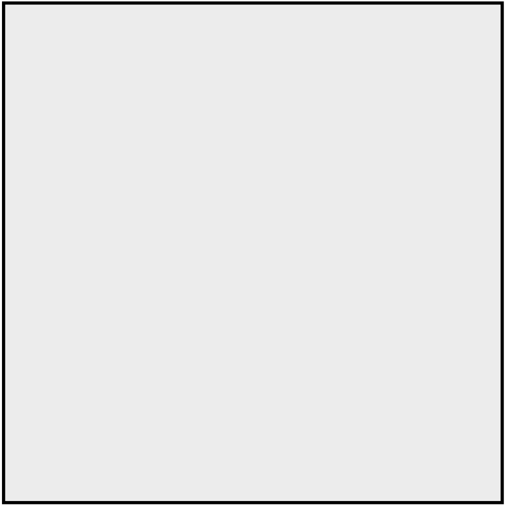

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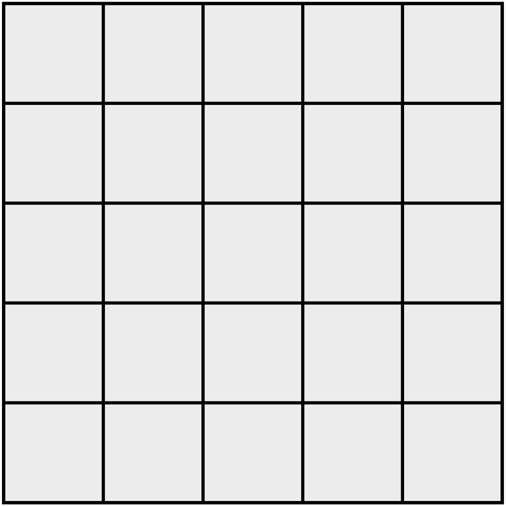

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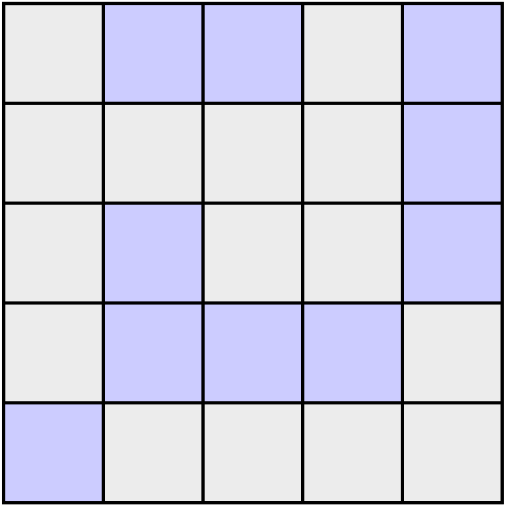

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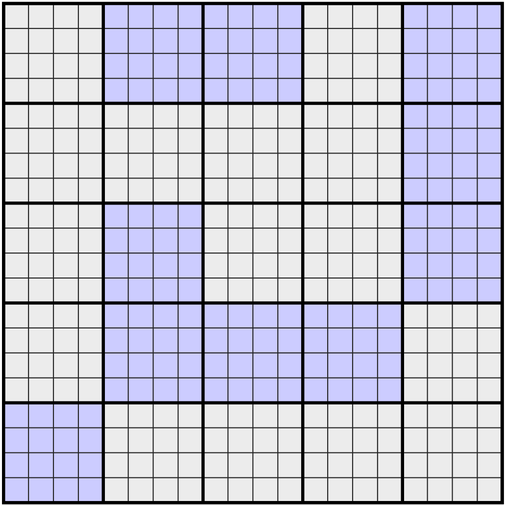

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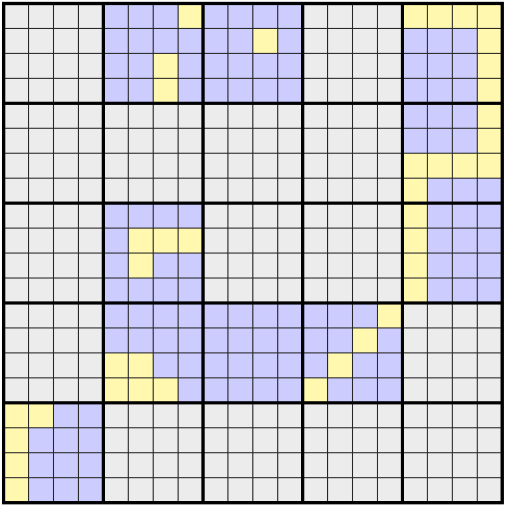

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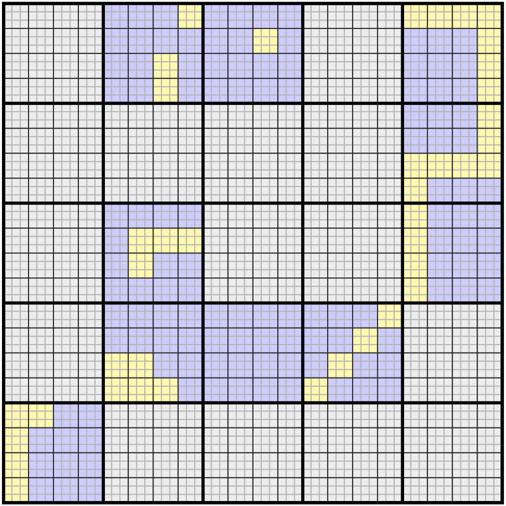

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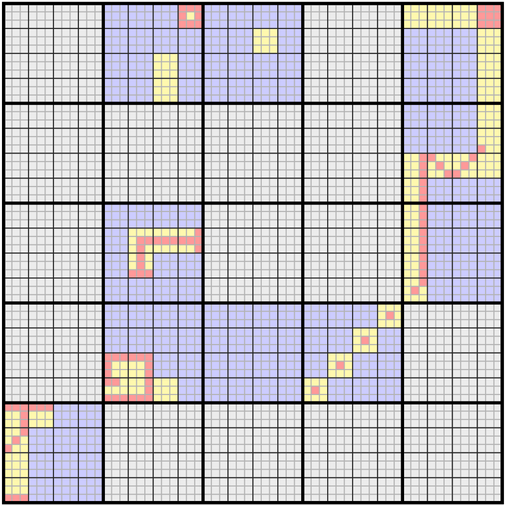

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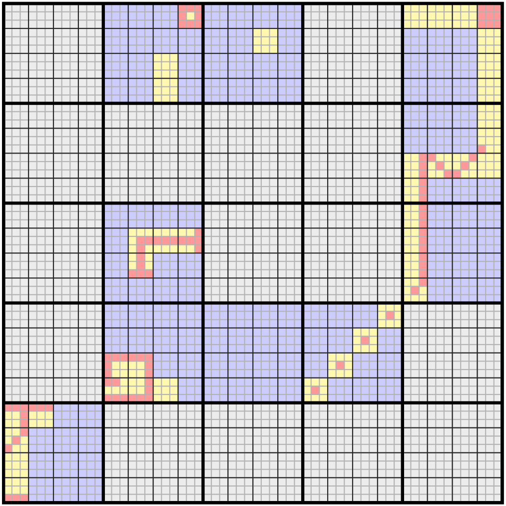

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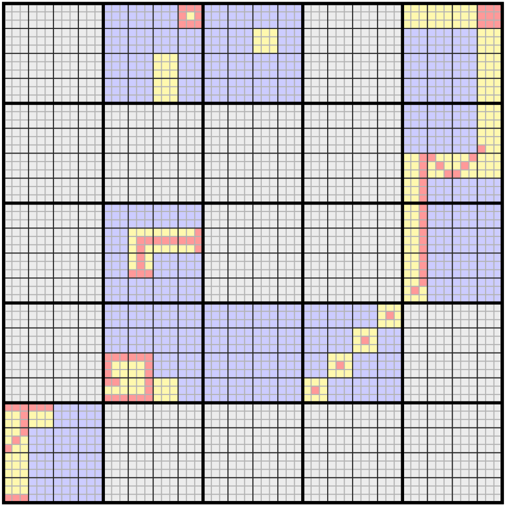

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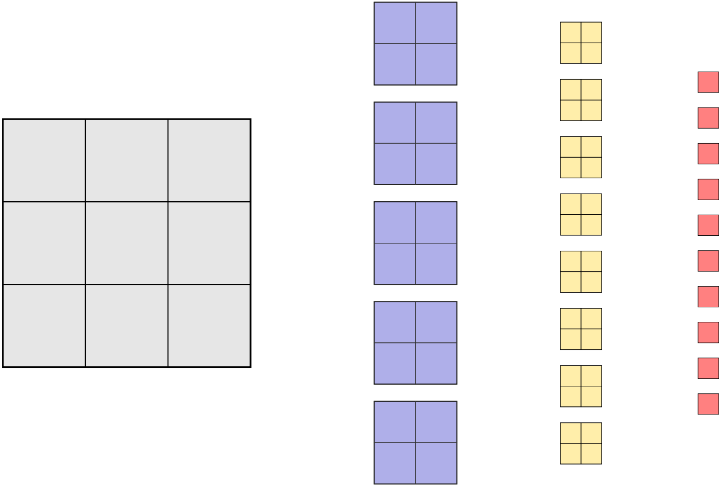

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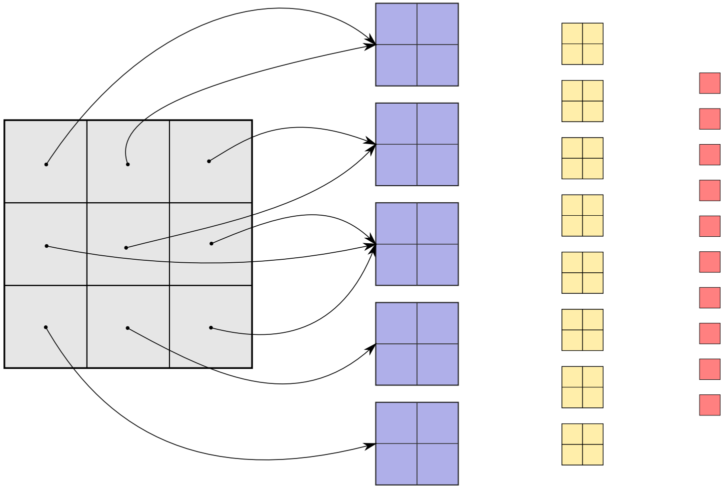

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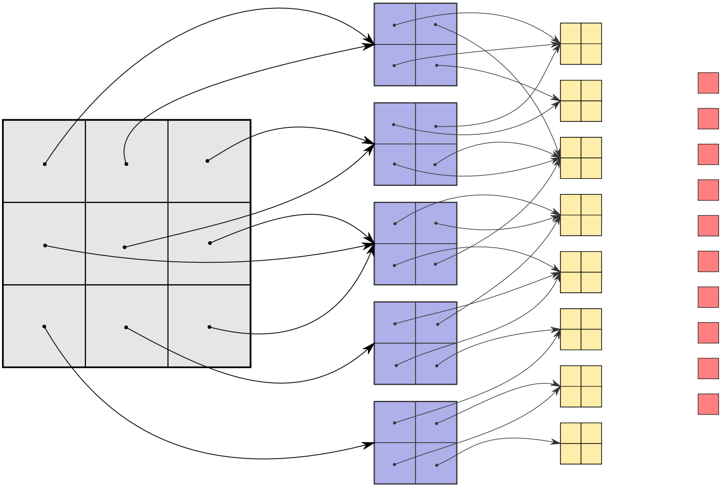

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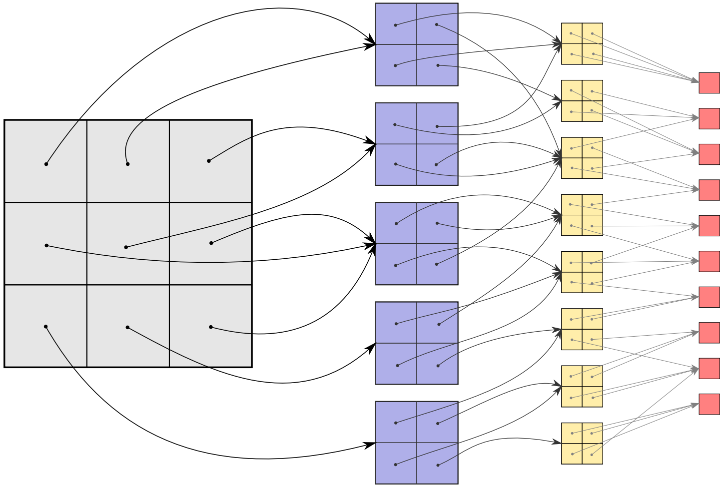

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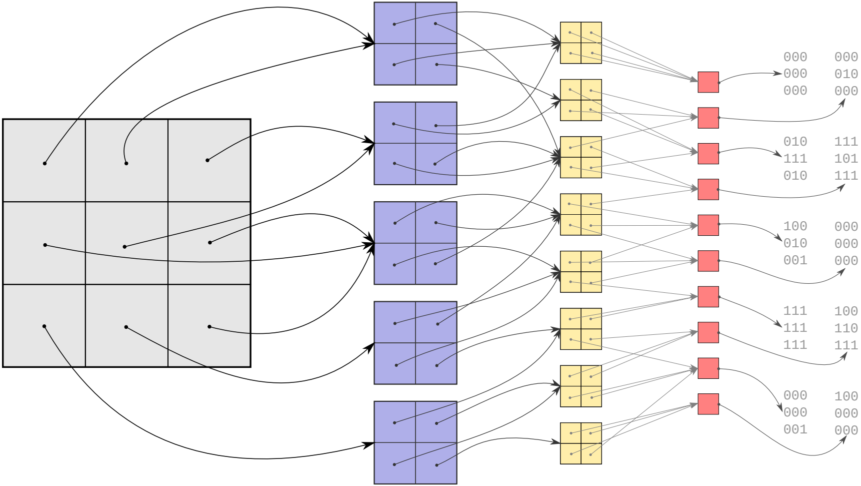
And store each unique zone of $\mathcal{D}_{\text{last}}$ explicitly.



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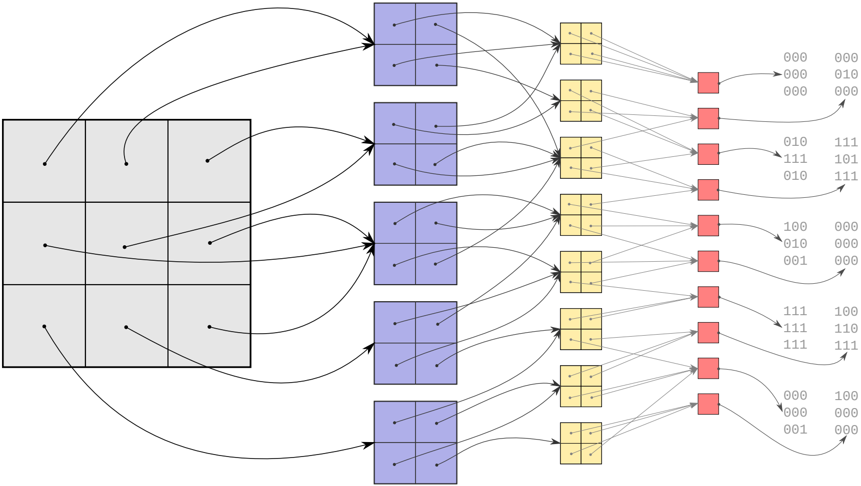

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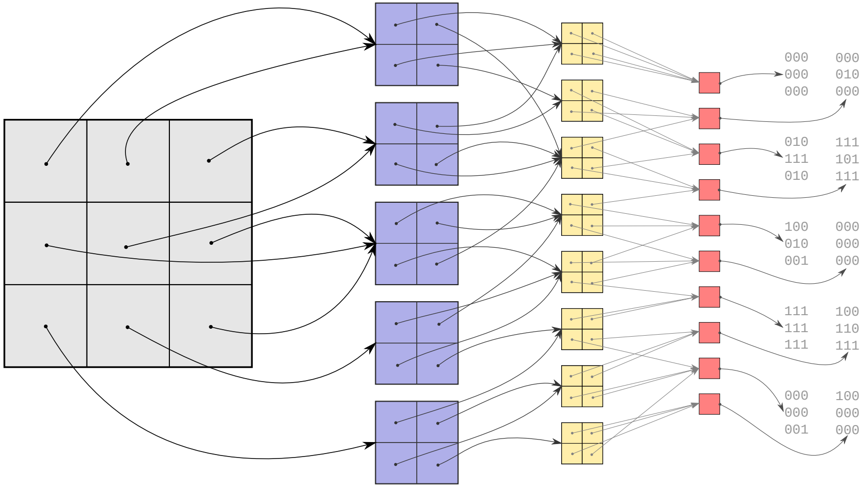

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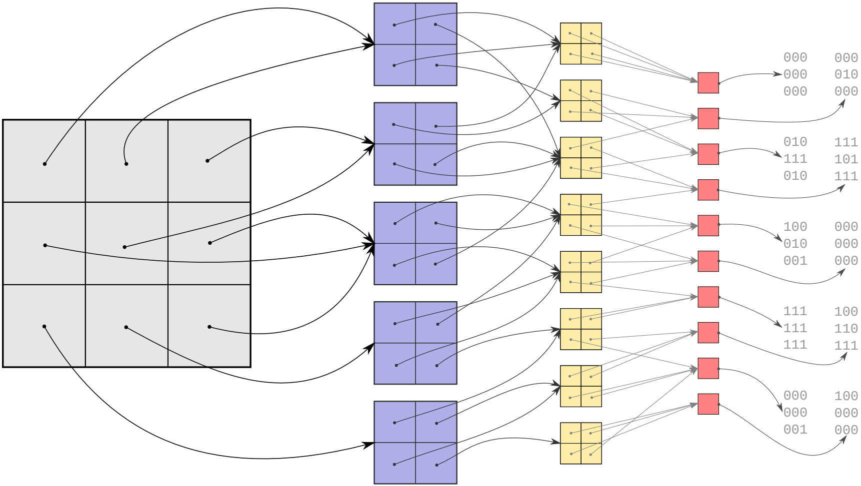
Time complexity: $\mathcal{O}(\text{last}) = \mathcal{O}(\log \log n)$.



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- The data structure needs to be modified slightly for **medium** zones.

Part 3

χ-boundedness

Pilipczuk, Sokołowski,
Graphs of Bounded Twin-Width are Quasi-Polynomially χ-Bounded

χ-boundedness

Let $\mathcal{C}$ – a hereditary class of graphs.

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Briański, Davies, Walczak (2022): **no.**

Twin-width and χ-boundedness

Theorem (Twin-width III)

Graphs of twin-width $\leq d$ are χ-bounded by $f_d(\omega) = (d+2)^{\omega-1}$.

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Note: the proof shows that

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Fix $d \in \mathbb{N}$. There exists a constant $\beta_d > 0$ such that graphs of twin-width $\leq d$ are χ-bounded by a **quasi-polynomial** function $f_d : \mathbb{N} \to \mathbb{N}$:

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Theorem (Bourneuf, Thomassé 2023)

Graphs of twin-width $\leq d$ are polynomially χ-bounded.

Ingredient: d-almost mixed minors

$$
\begin{array}{cc|ccc|cccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Corollary

$$\mathrm{tww}(G) \le d \implies G \text{ has a } (2d+2)\text{-mixed free adjacency matrix}$$
$$\implies G \text{ has a } (4d+4)\text{-almost mixed free adjacency matrix.}$$

Idea

If we had

$$f_d(\omega) \leq \mathsf{const}(d) \cdot f_d(0.9\,\omega),$$

then we would get $f_d(\omega) = \mathrm{poly}(\omega)$!

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Induction on $\omega(G) \geq 1$.

G – a graph,
M – a d-almost mixed free adjacency matrix of G,
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Partition $V(G)$ into intervals $A_1 \cup A_2 \cup \cdots \cup A_k$ (**blobs**) so that

$$\omega(G[A_i]) = 0.9\,\omega \qquad \text{for } i = 1, 2, \ldots, k.$$

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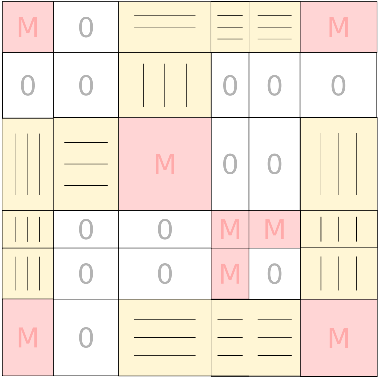
$\mathcal{D} :=$ a (symmetric) division of M from the partition $A_1, \ldots, A_k$.



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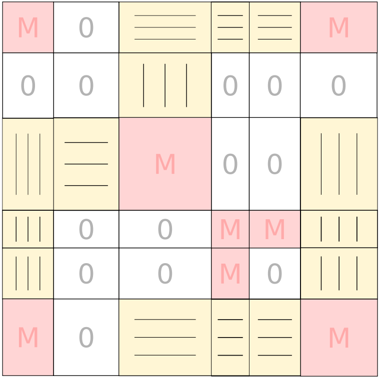

First, paint each **blob**

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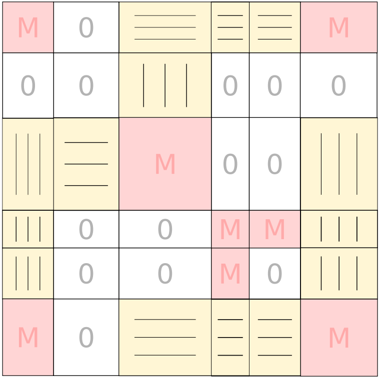

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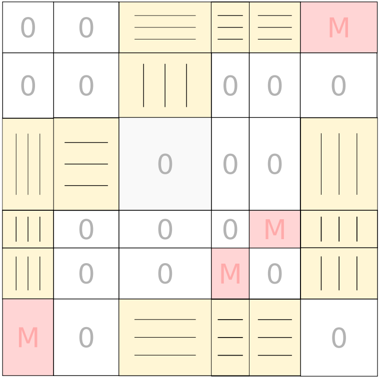

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For each color class C: each intersection $C \cap A_i$ is an **independent set**!

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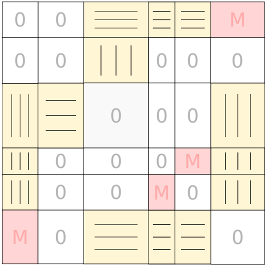

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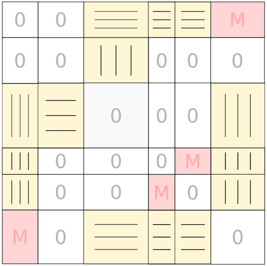

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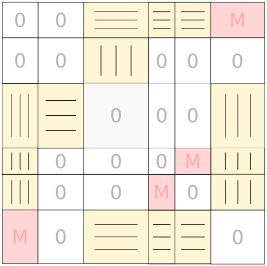

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$$\chi(G[C]) \le \underbrace{\mathrm{const}(d)}_{\text{remove } \boxed{\text{M}} \text{ with Marcus–Tardos}} \cdot \underbrace{f_d(0.2\omega)}_{\text{afterwards, clique number } \le 0.2\omega}$$

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- Even worse than exponential...

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Rich blobs

Fix a blob B_i ($i < k$), and let $B_i = \{\ell_i, \ell_i + 1, \dots, r_i\}$. We call B_i **rich** if:

- every vertex $v \in B_i$ is adjacent to any $s \in \{r_i + 1, r_i + 2, \dots, n\}$,
- no two consecutive vertices of B_i are twins with respect to $\{r_i + 1, r_i + 2, \dots, n\}$.

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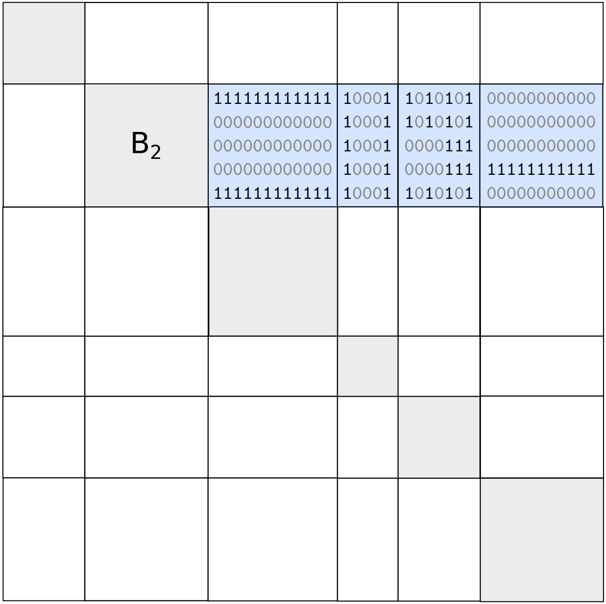

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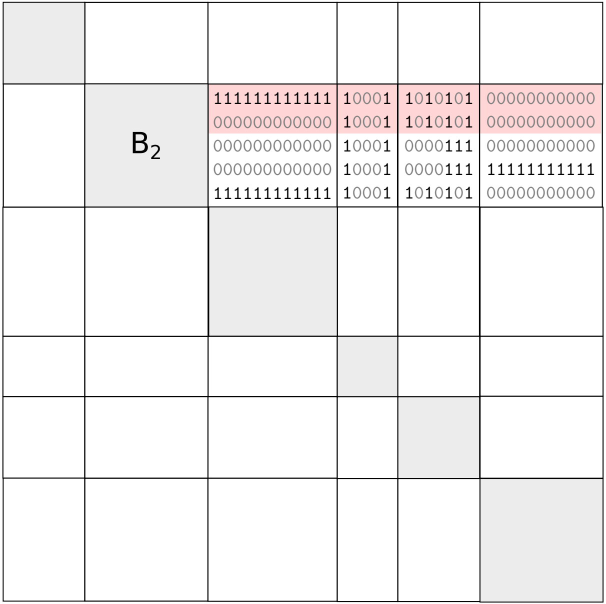

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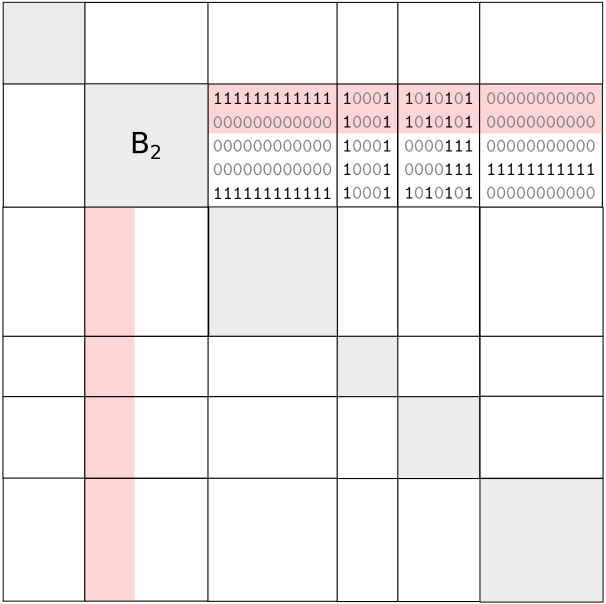

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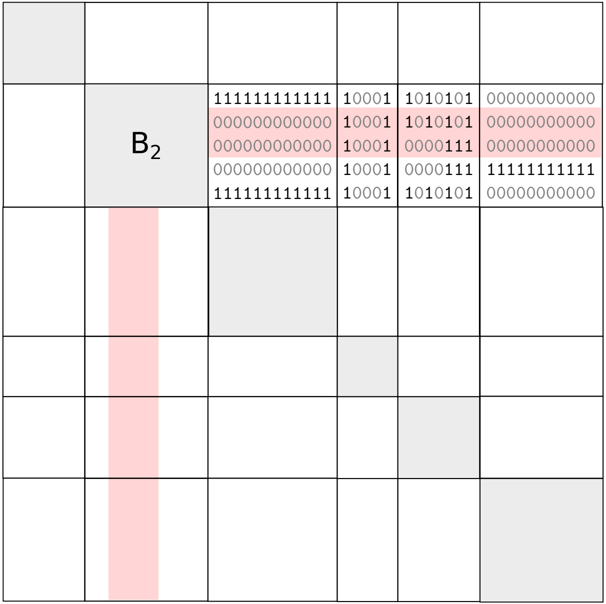

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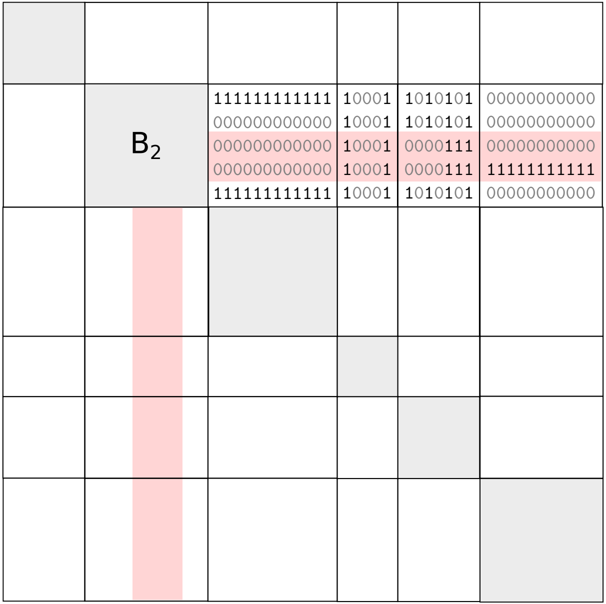

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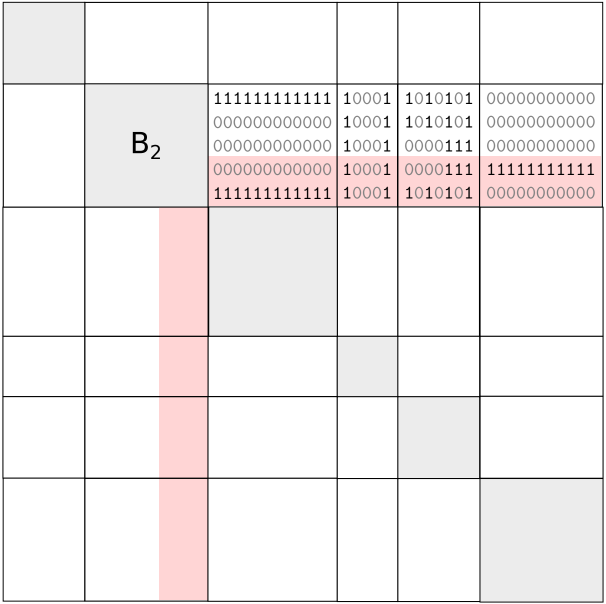

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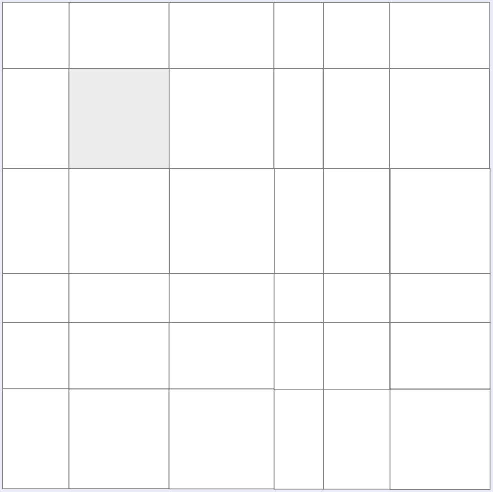

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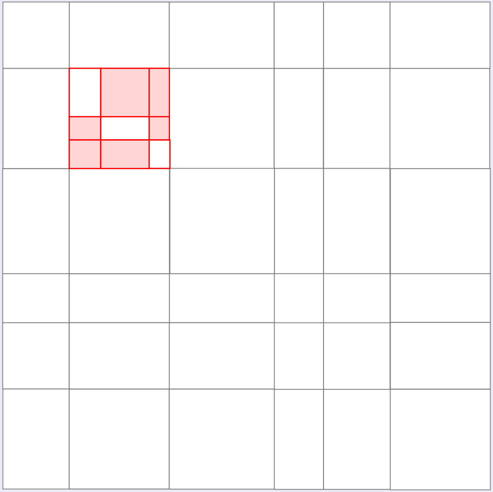

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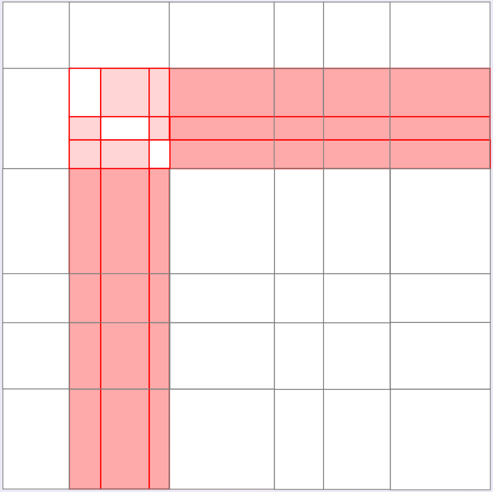

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By induction: quasi-polynomial!

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- Quasi-polynomial again!

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Subexponential ($2^{O(\omega^\varepsilon)}$ for any $\varepsilon > 0$ if parameters chosen carefully).

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We eventually get:

$$f_d(\omega) \leq \text{const}(d) \cdot \left\{ f_d(0.9\,\omega) + f_{d-1}(\omega^d)^2 \cdot \sum_{u=0}^{\lfloor \log_2(0.1\,\omega) \rfloor} f_d(2^{u+1}) \cdot f_d\left(\frac{0.2\,\omega}{2^u} + 1\right) \right\}$$

This resolves to $f_d(\omega) = 2^{\beta_d \cdot \log^d \omega}$.

Thank you!