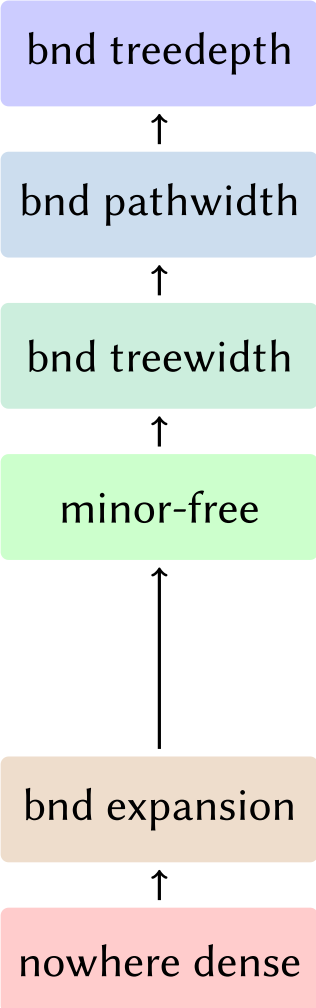
Stable graphs of bounded twin-width

joint work with Jakub Gajarský and Szymon Toruńczyk

1st Twin-width Workshop

Aussois, May 24th, 2023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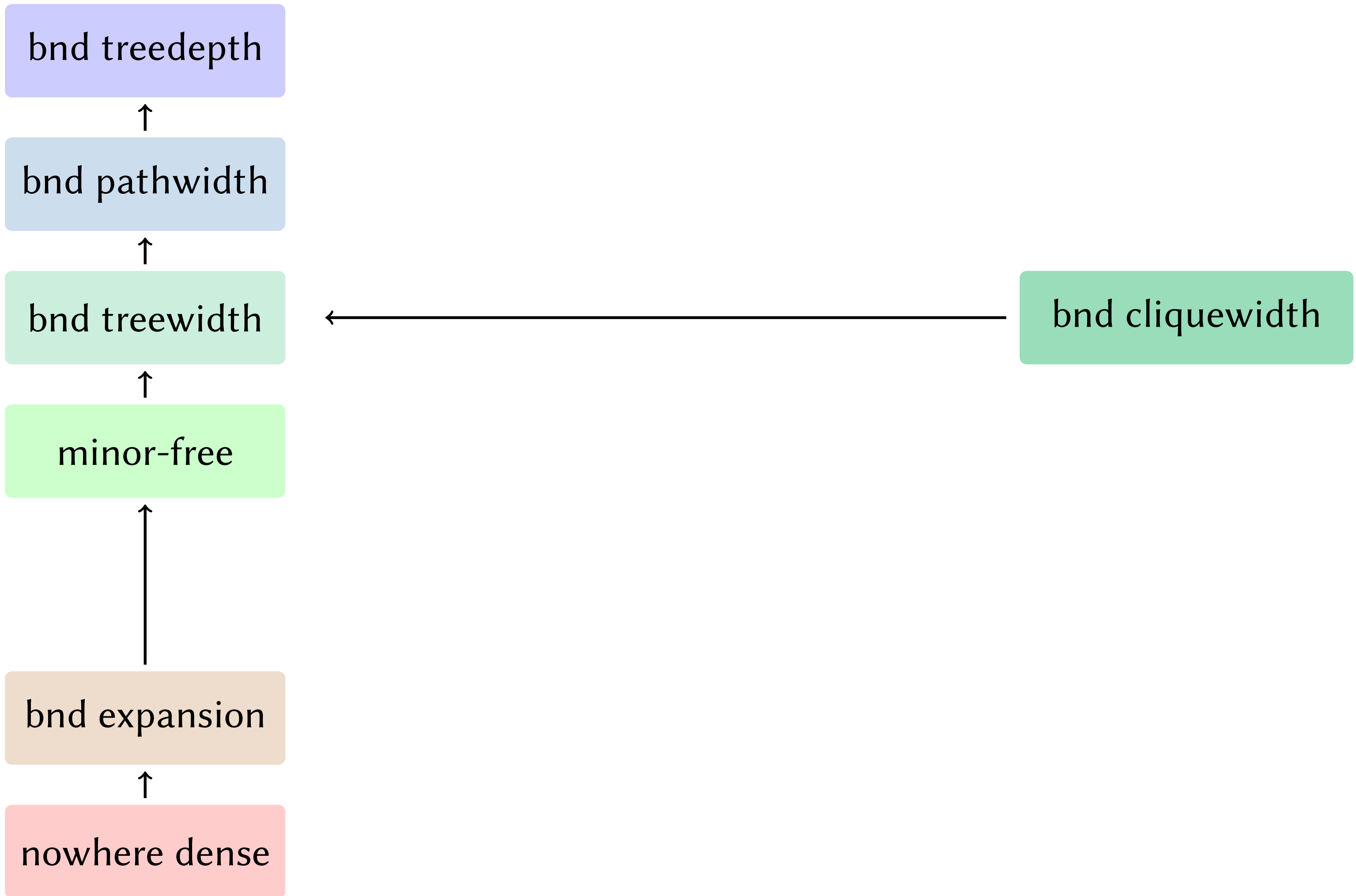
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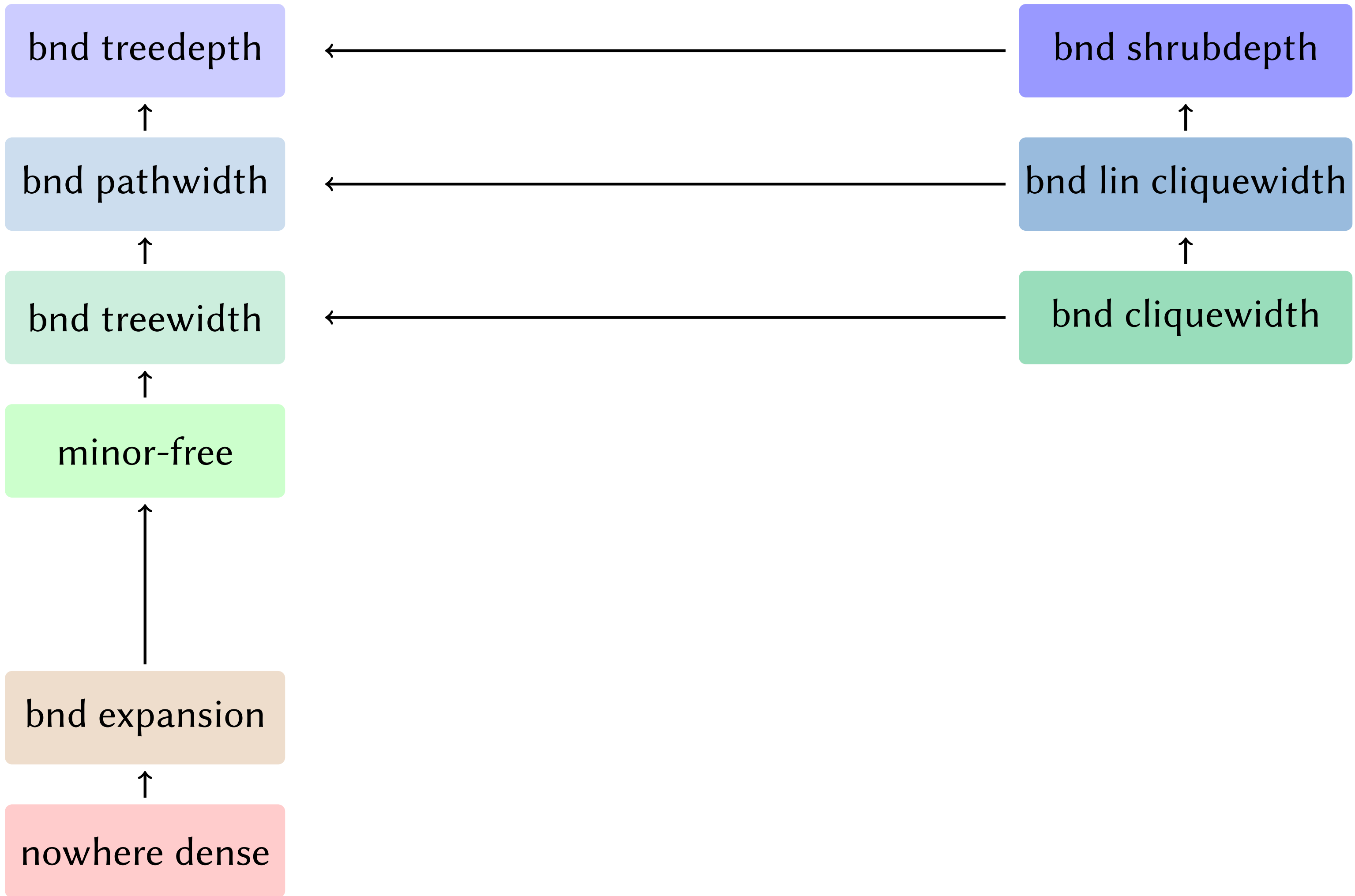
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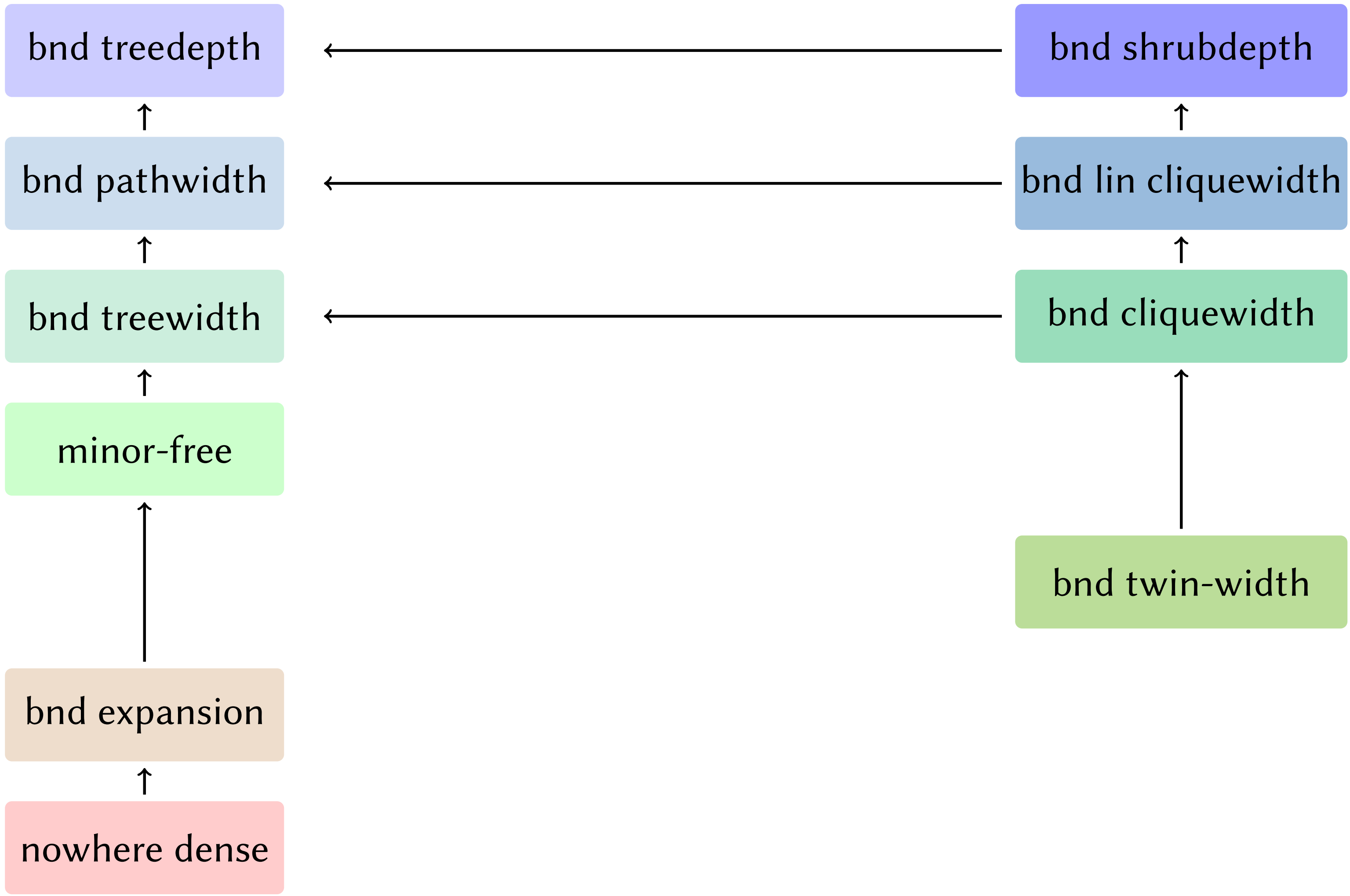
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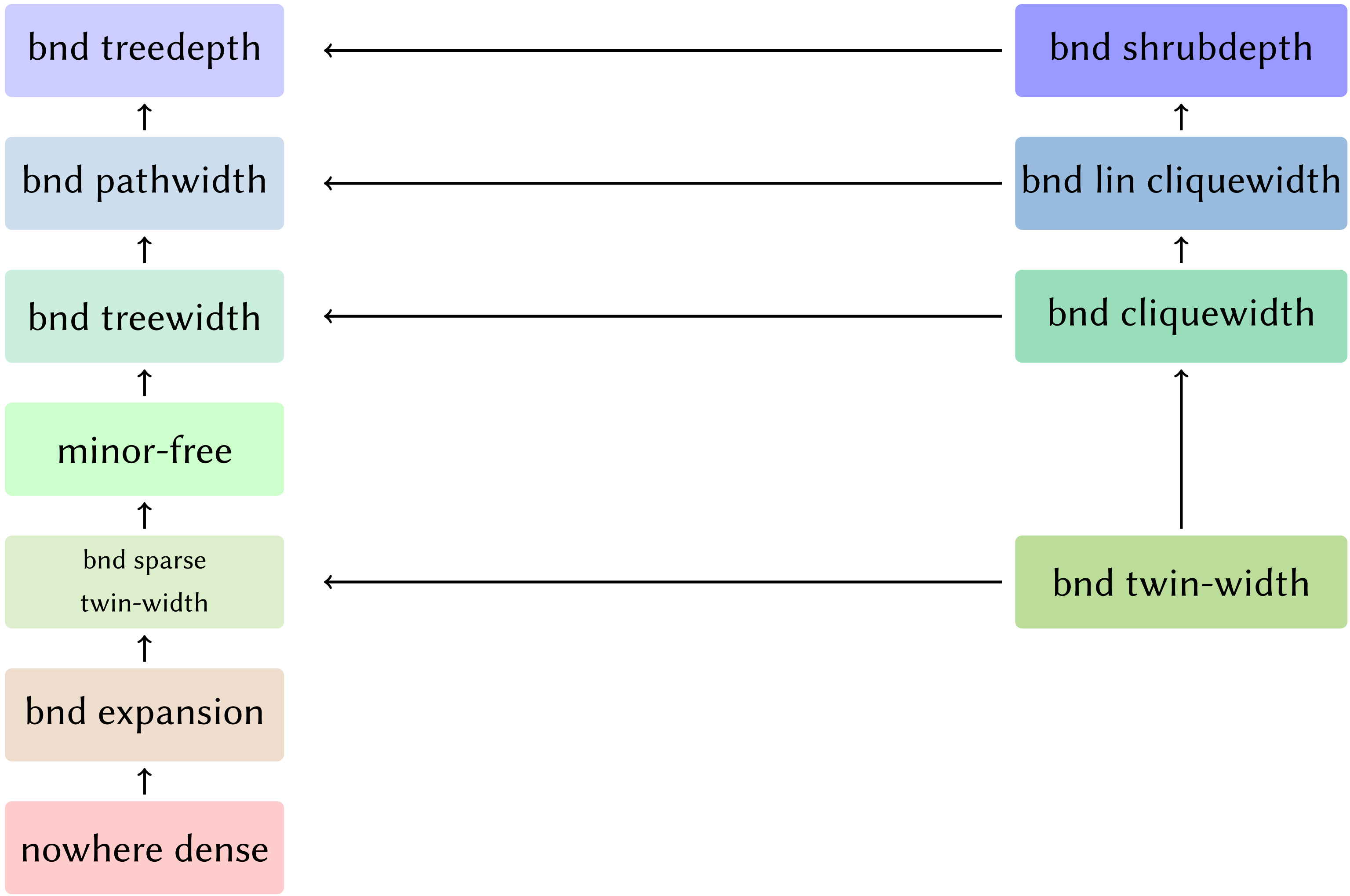
Also: mixed minors $\rightsquigarrow$ grid minors.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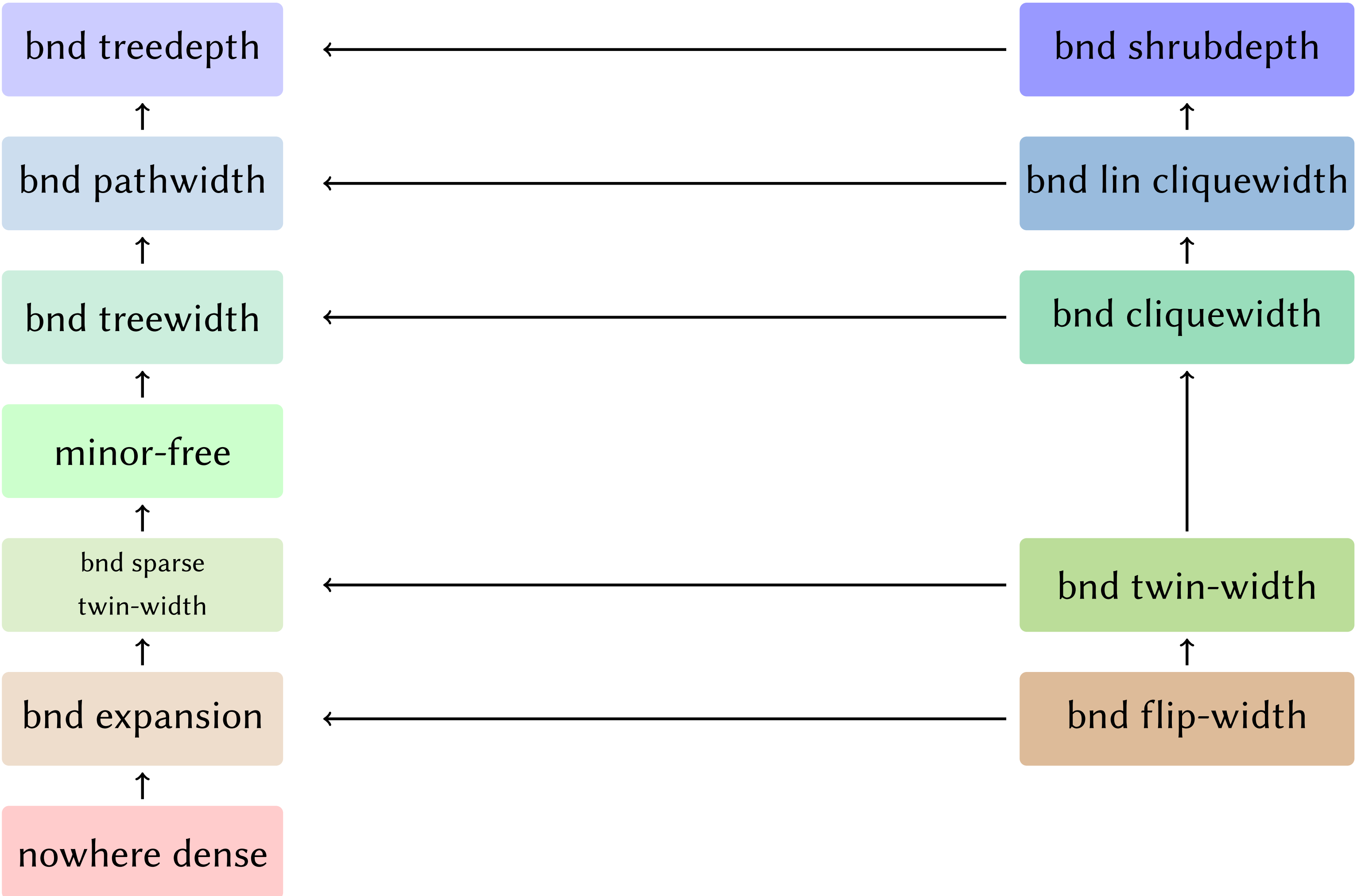
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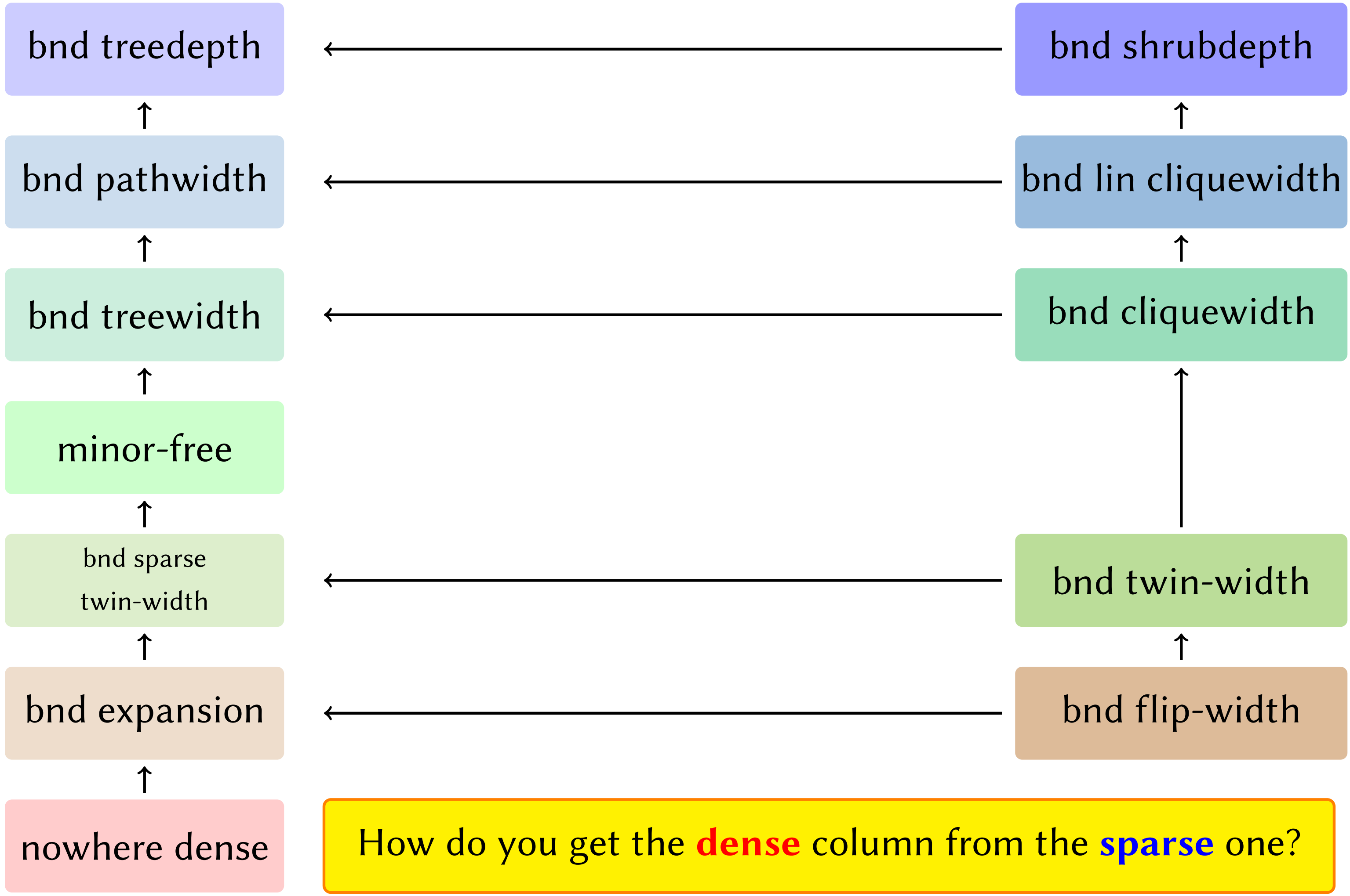
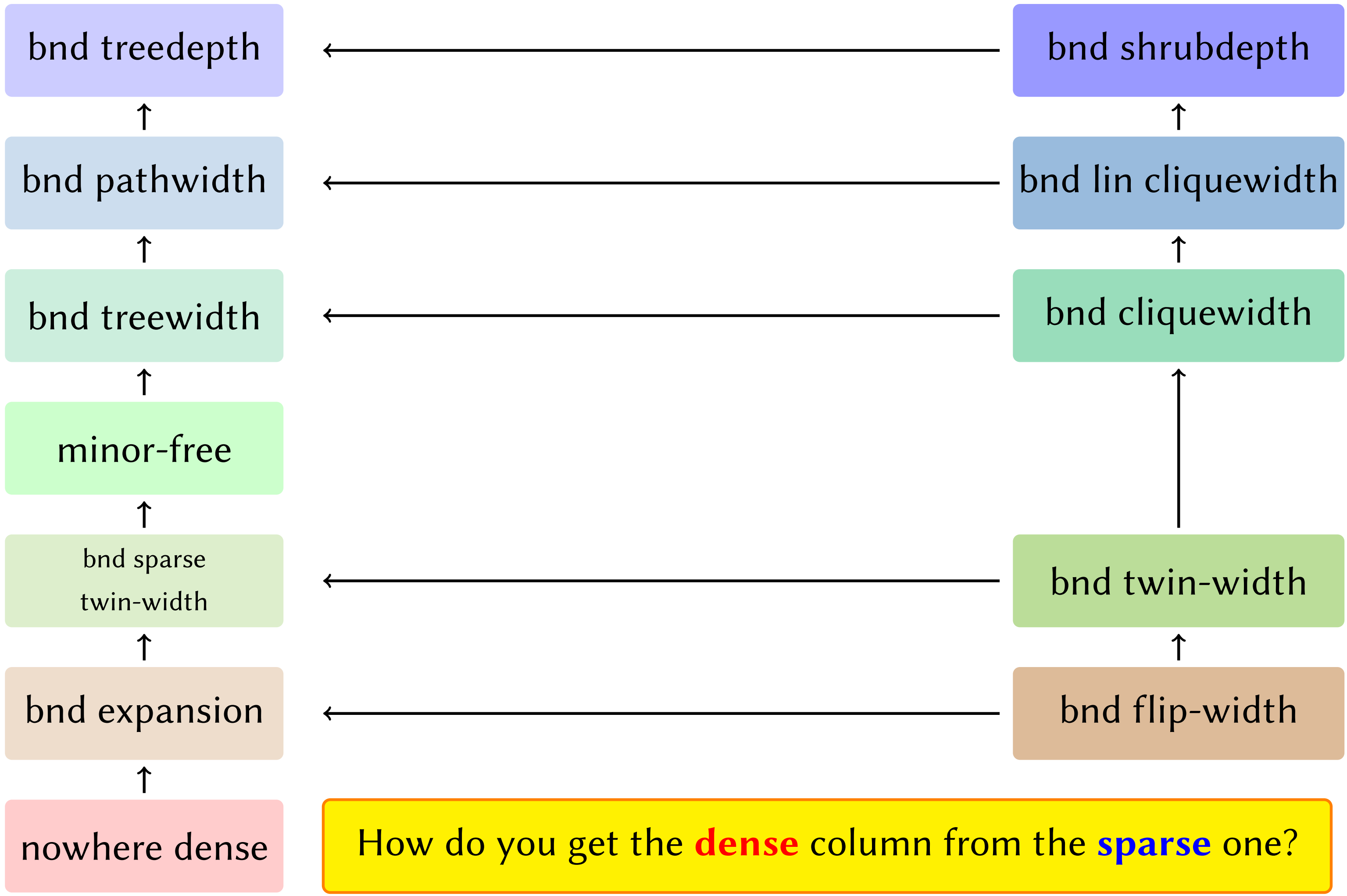


How do you get the **dense** column from the **sparse** one?



Idea: Close under logically defined operations.

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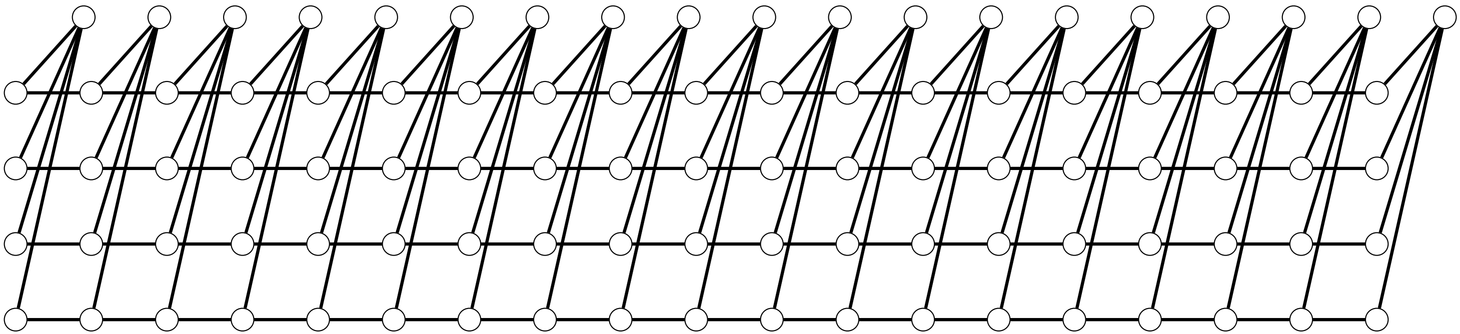

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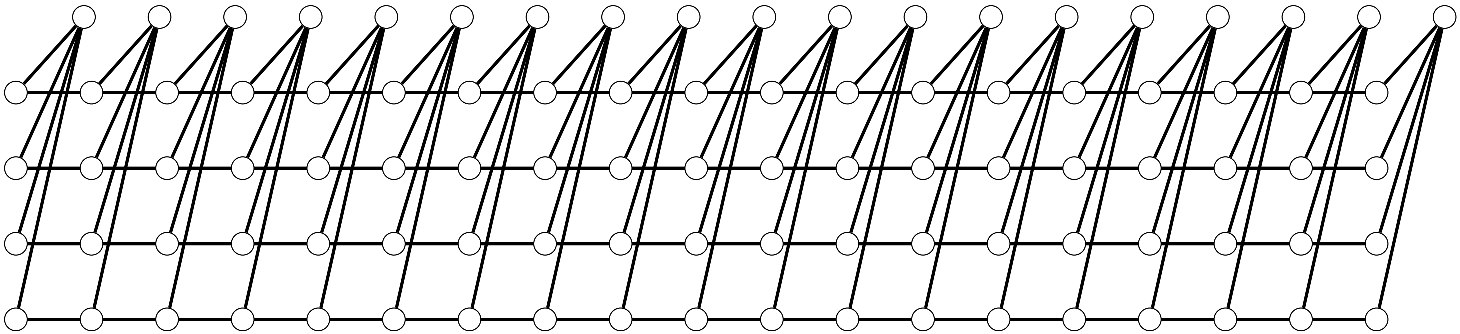

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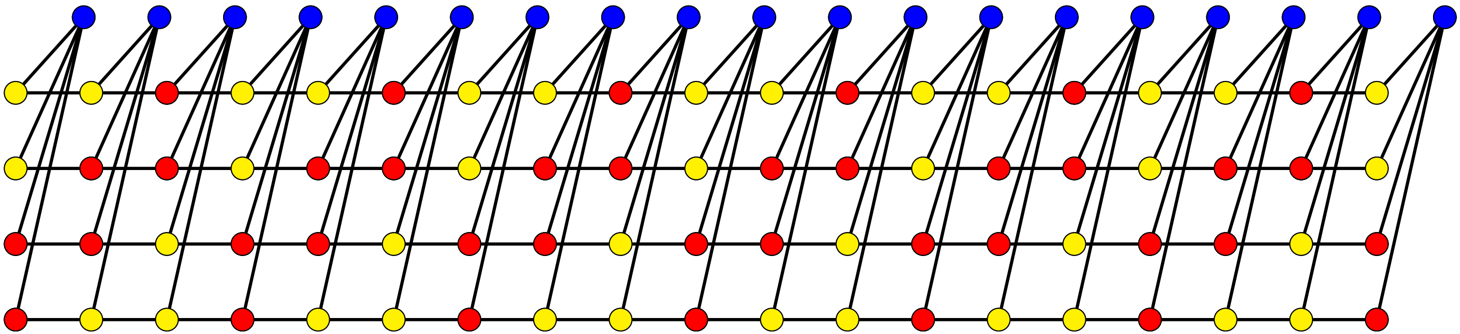

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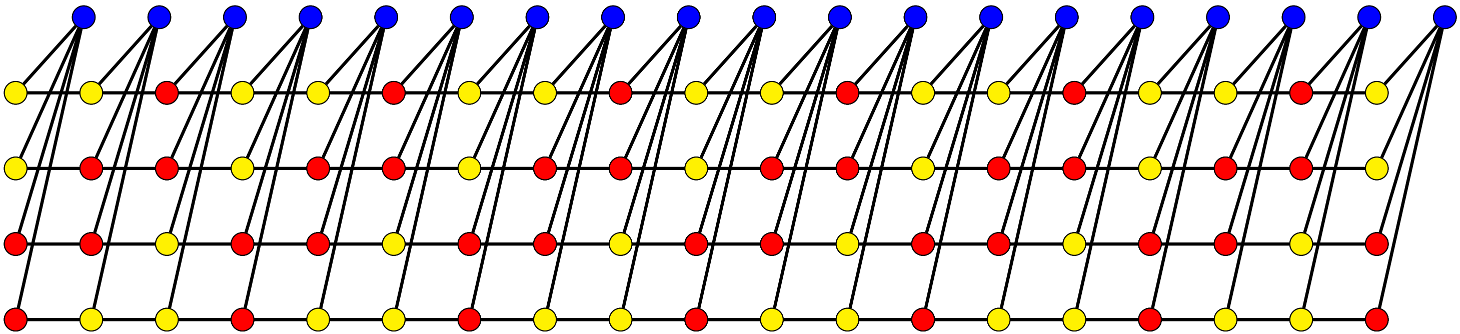

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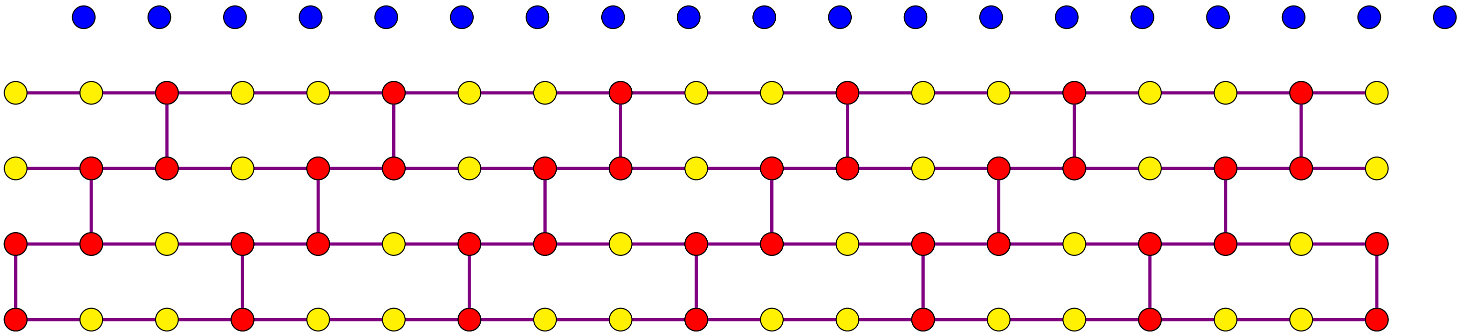

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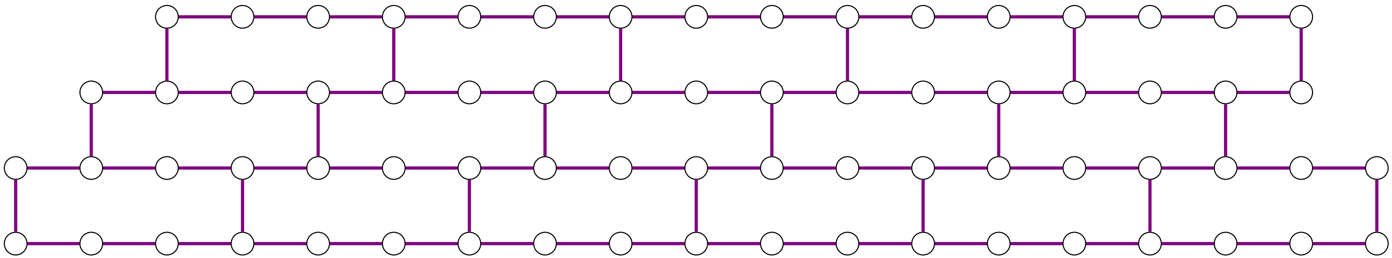

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Step 3: Take any induced subgraph.

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FO-transductions, MSO_1-transductions, MSO_2-transductions, ...

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Equivalently: **bnd cliquewidth** = **structurally bnd treewidth**?

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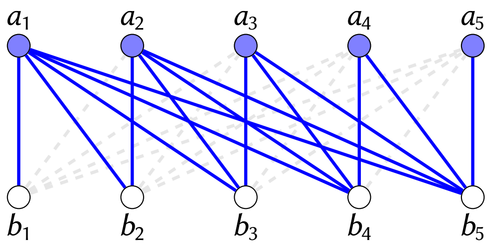

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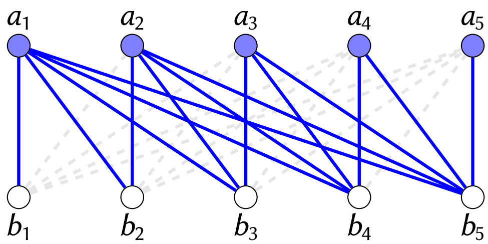

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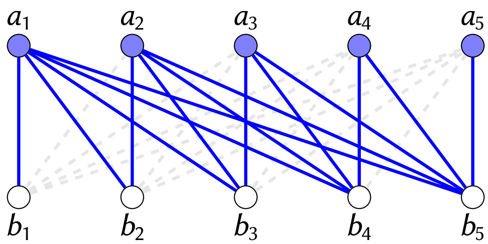

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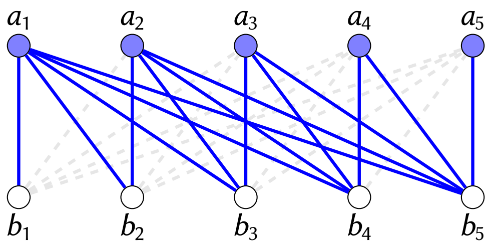

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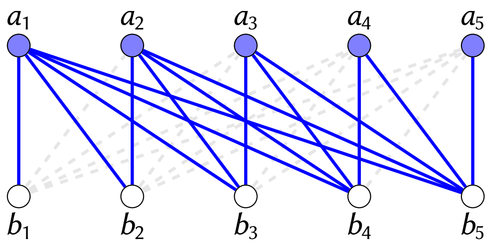

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Intuition: Whatever we transduce from sparse classes, no half-graphs.

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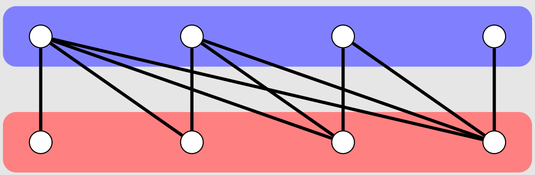
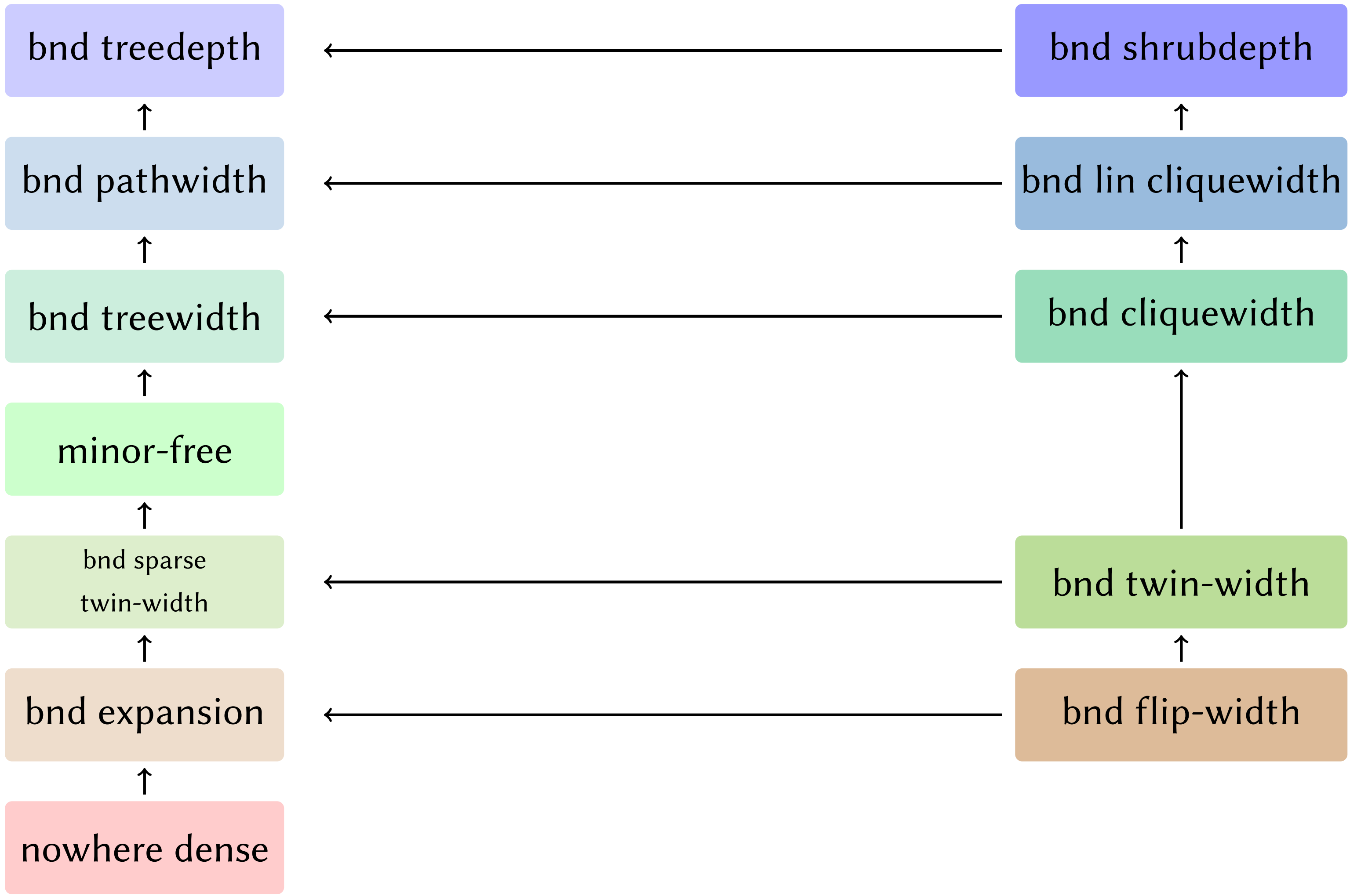
Fact: If $\mathscr{C}$ is **mon dependent**, then

$\mathscr{C}$ is **mon stable** $\Leftrightarrow$ $\mathscr{C}$ has a **stable edge relation**;

this means excluding some **semi-induced** half-graph.

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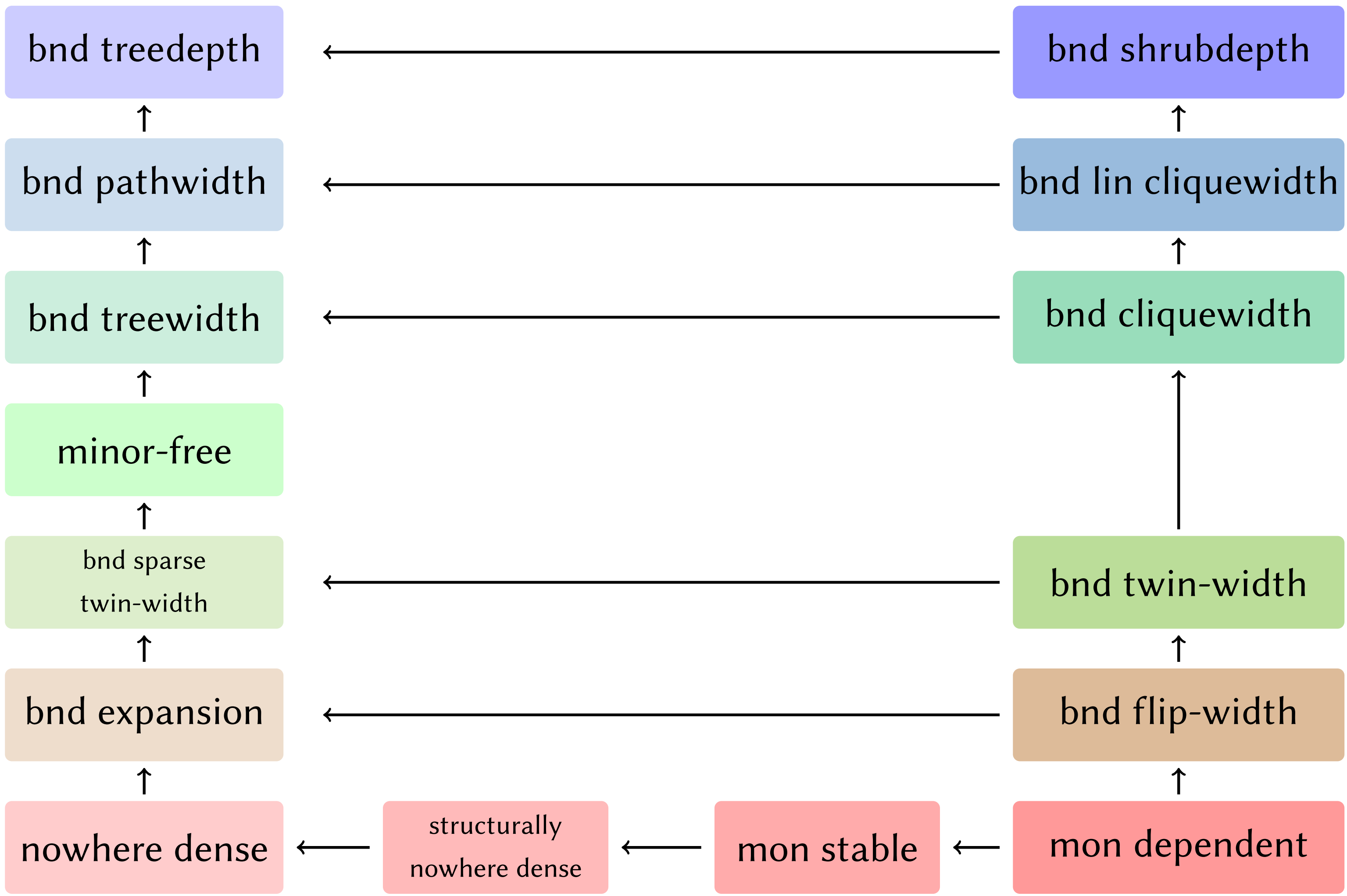
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mon stable
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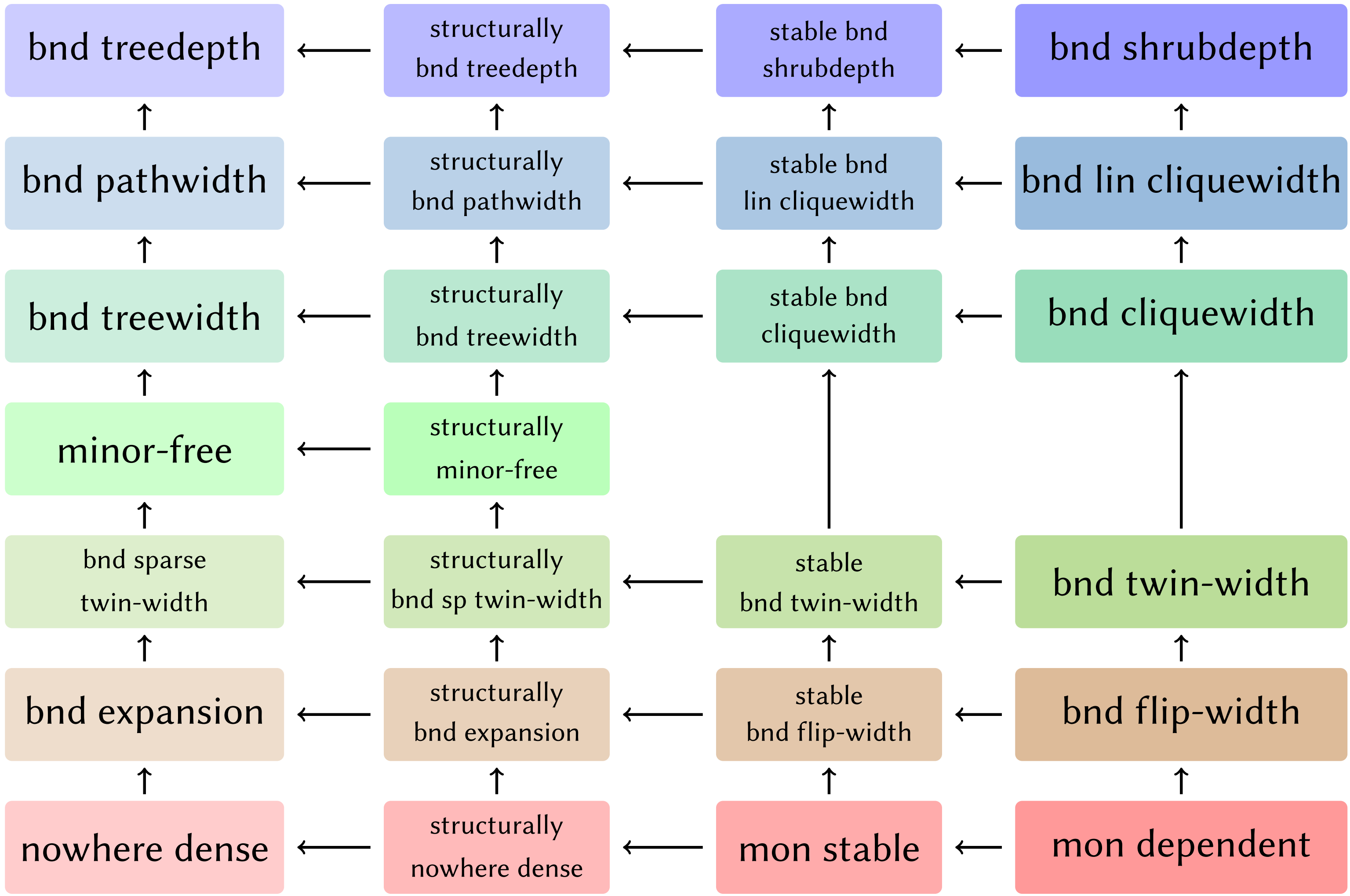
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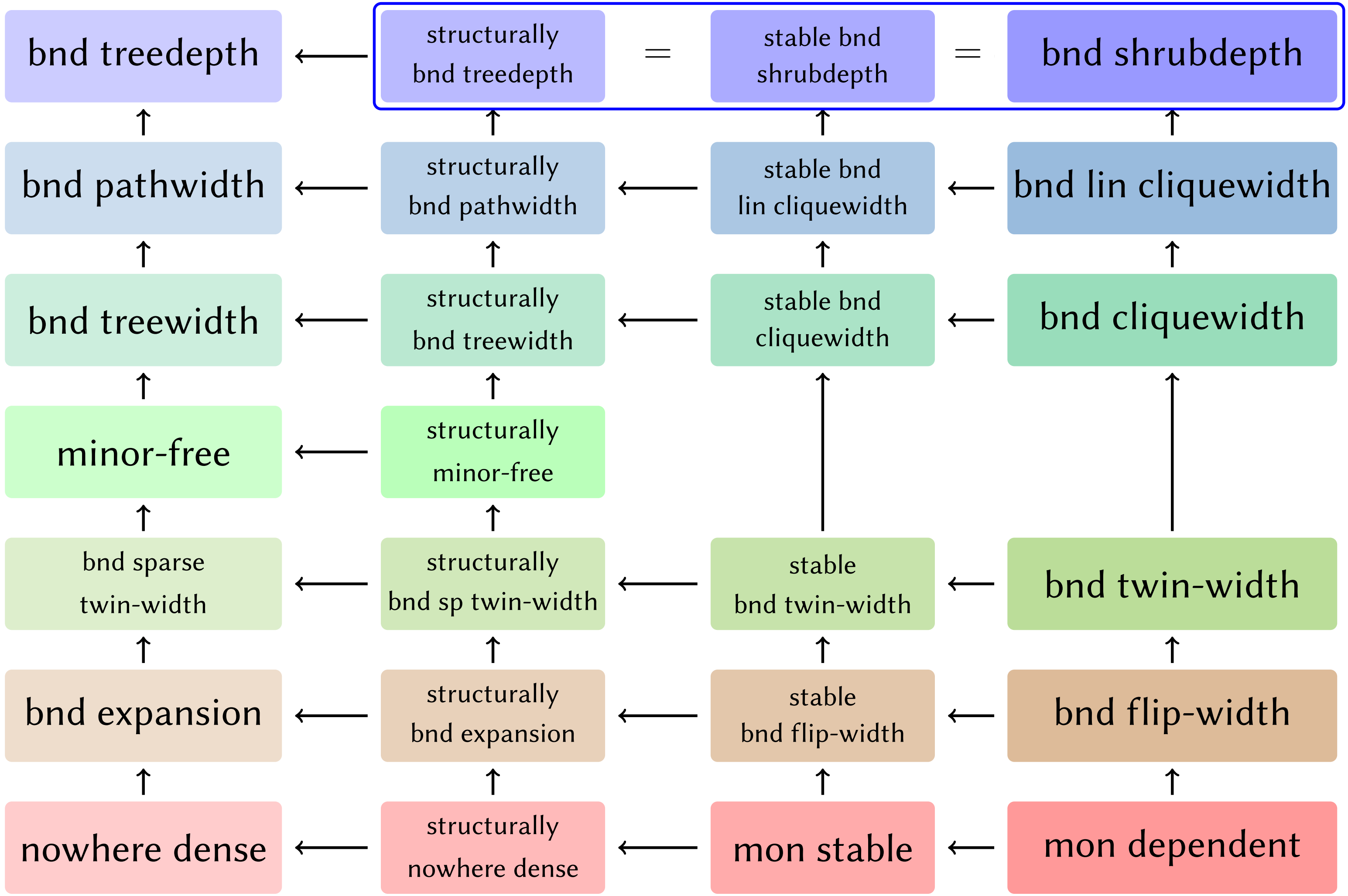
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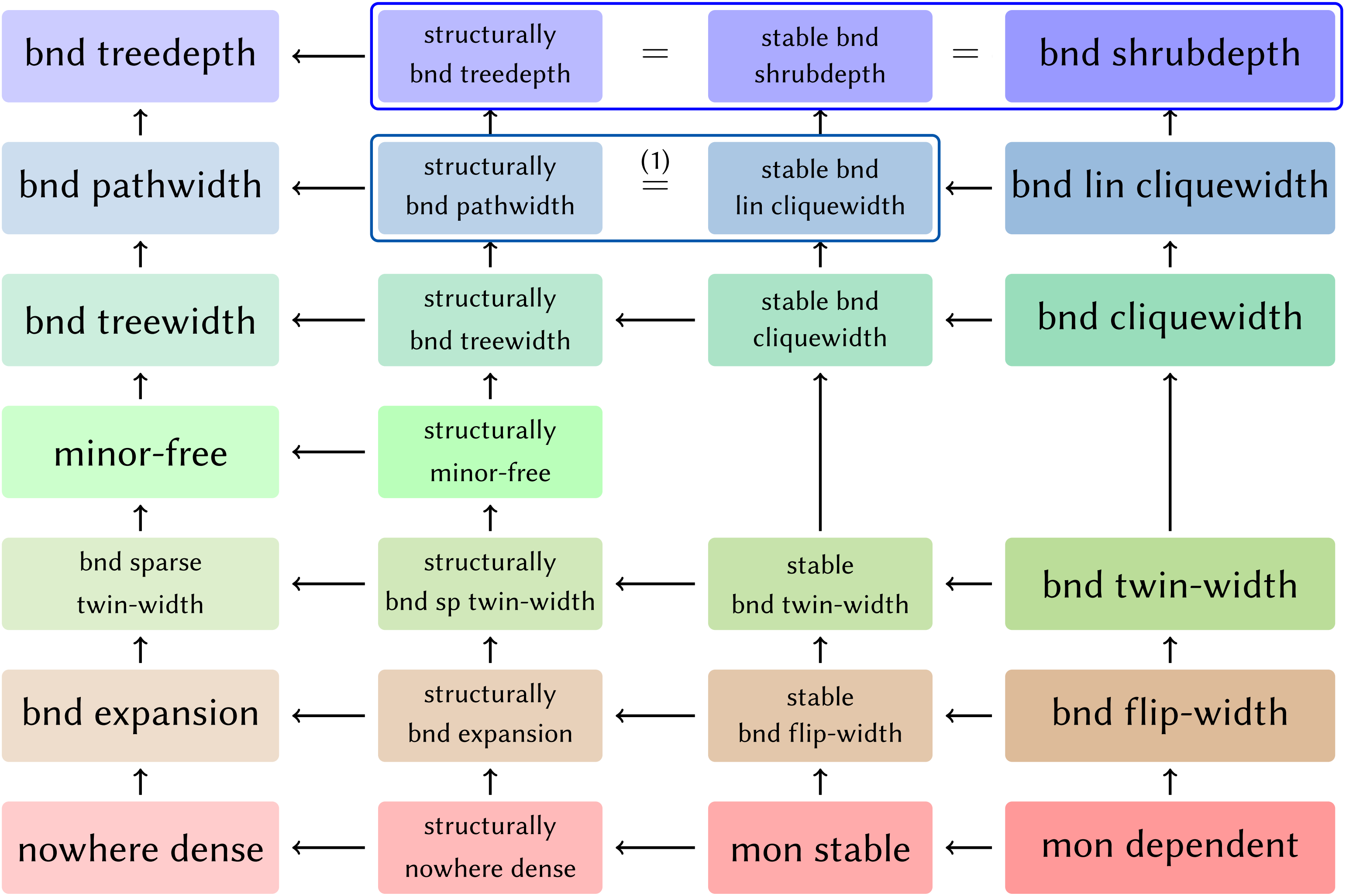
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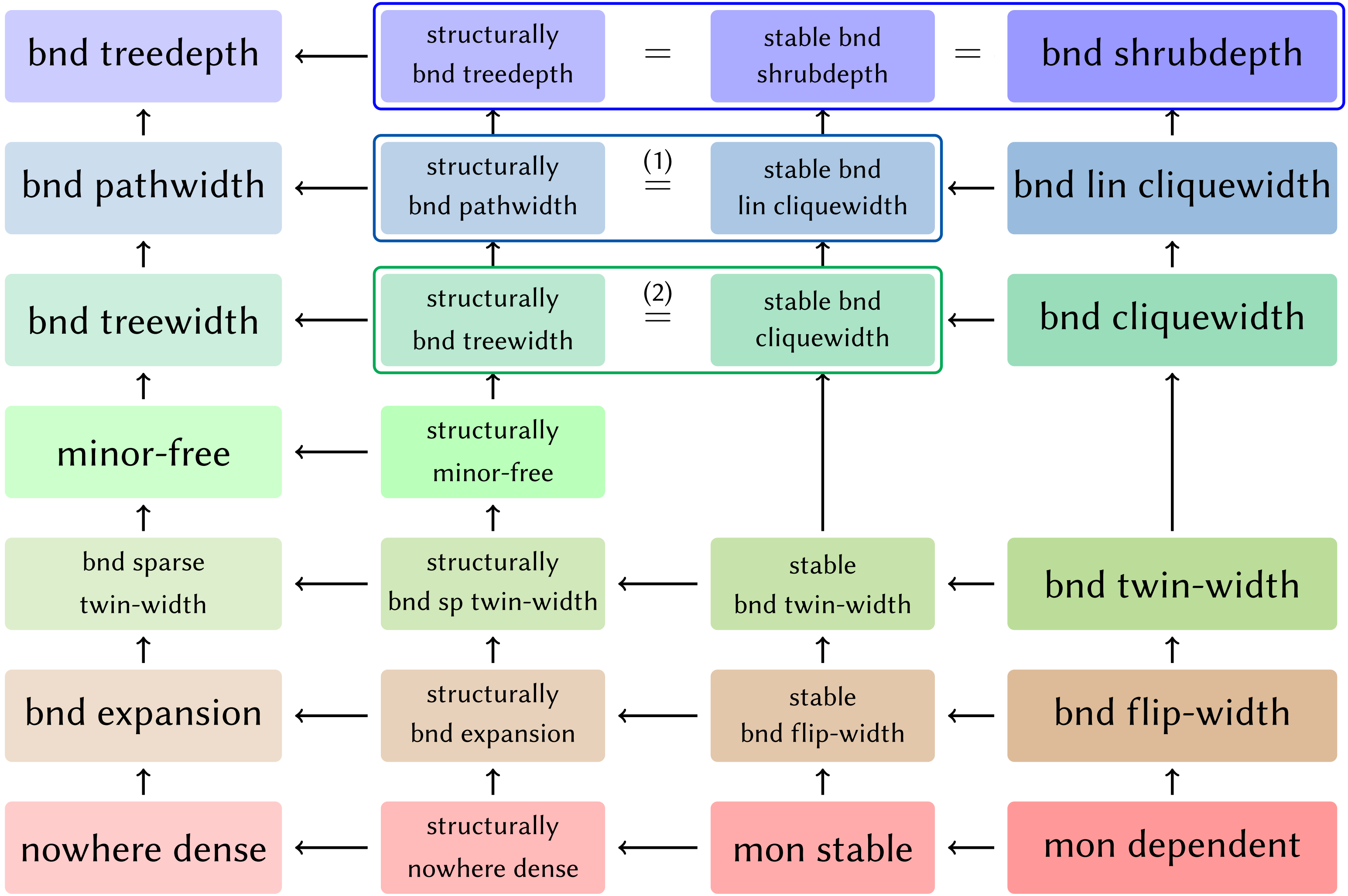

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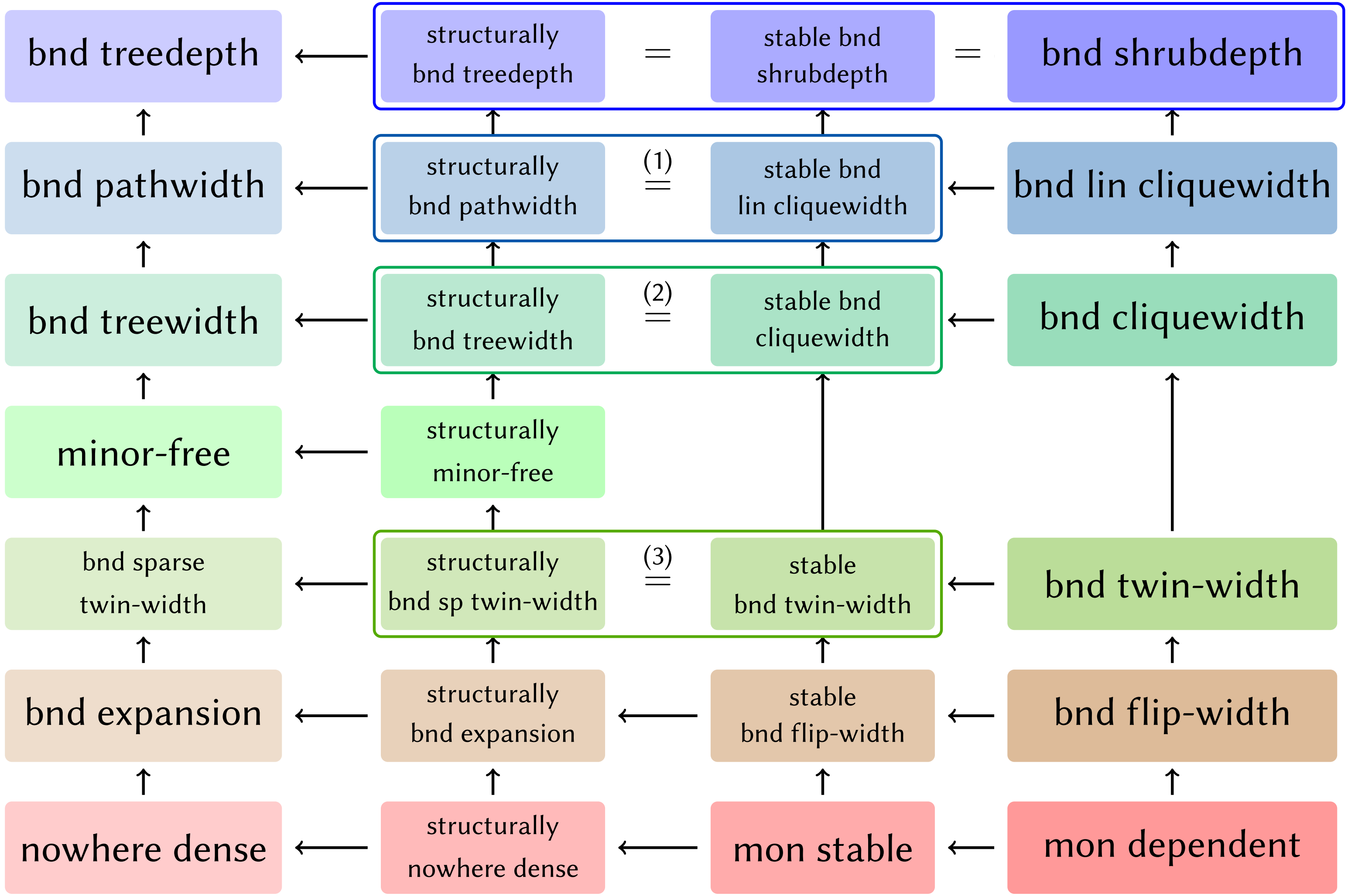

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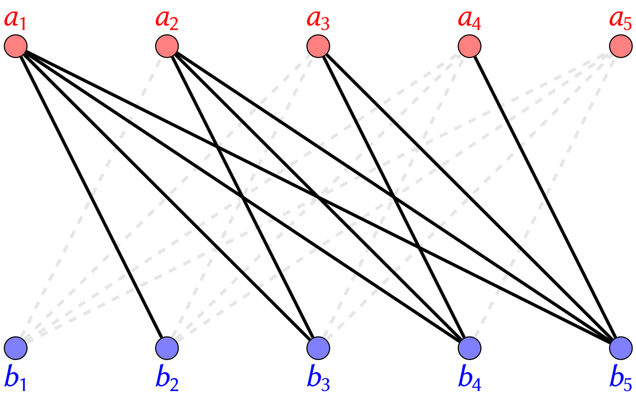

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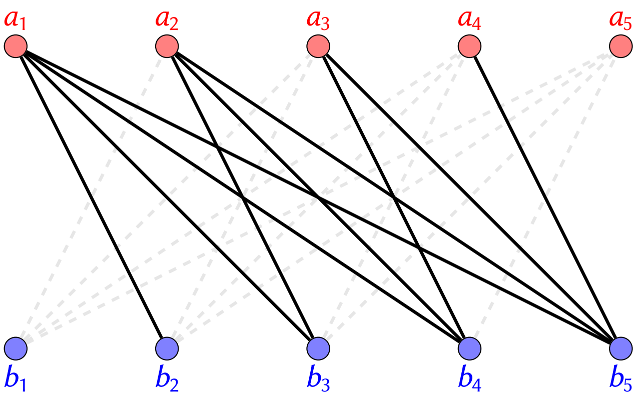

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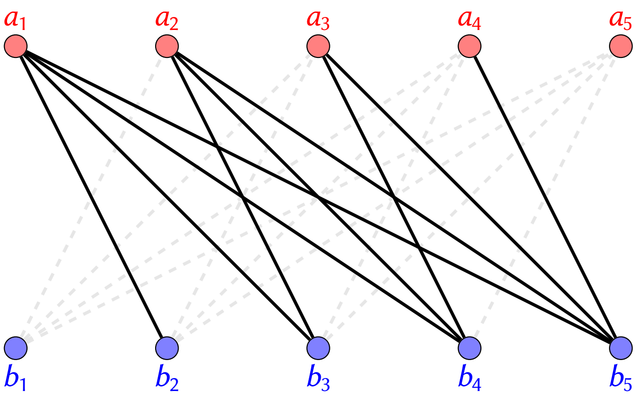

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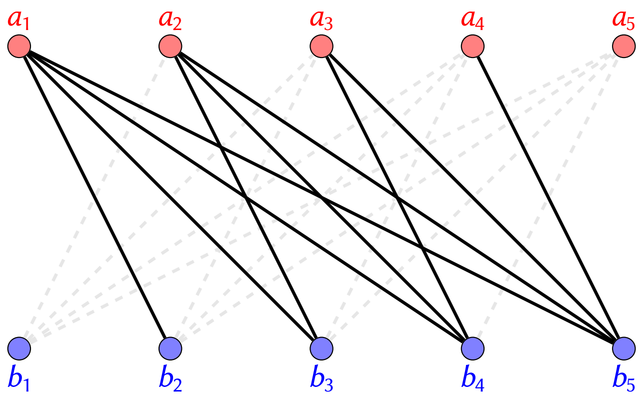

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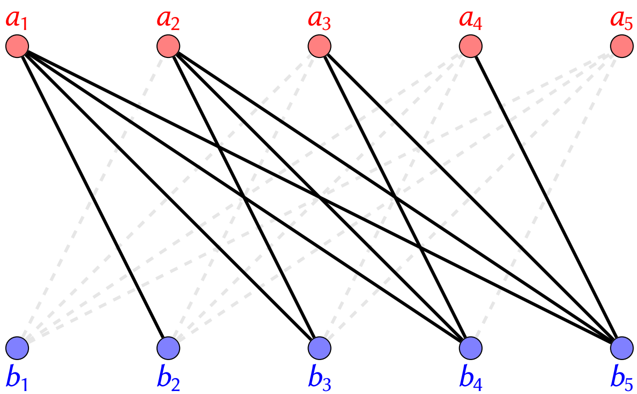

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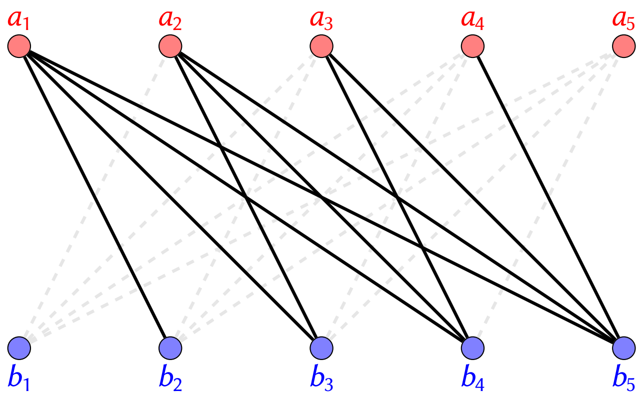

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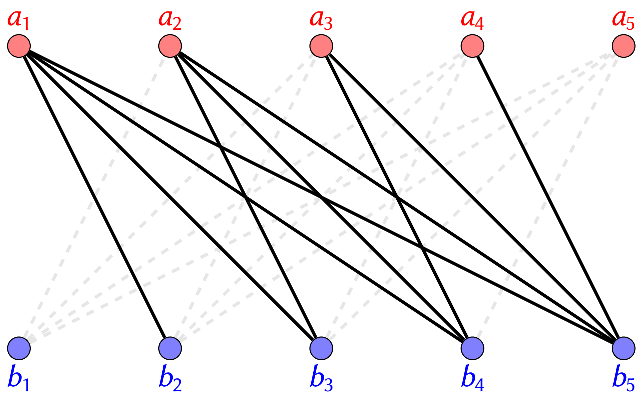

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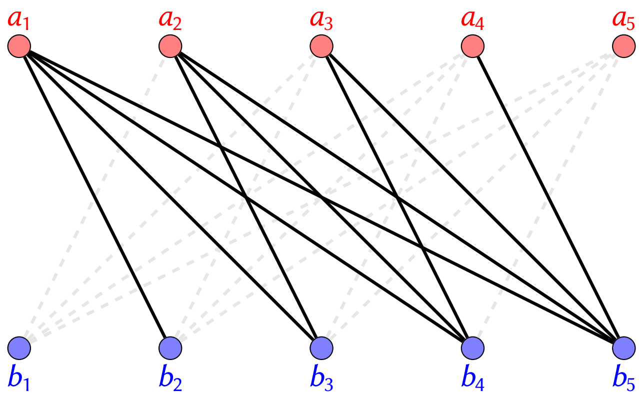

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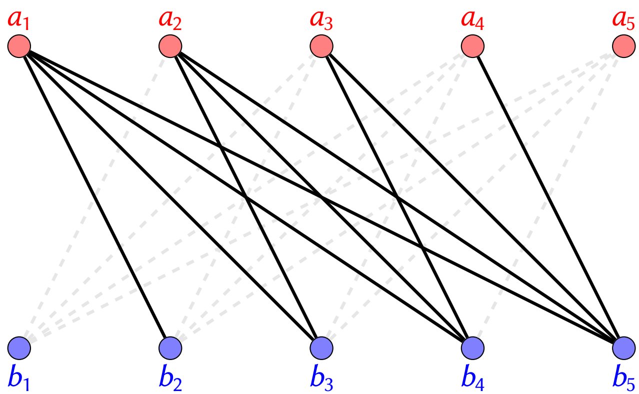

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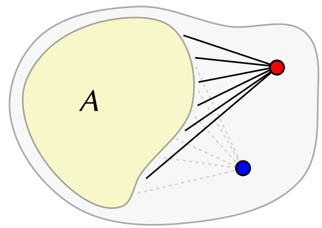

Idea: Use index as a **progress measure**.

Obs: If $A \subseteq V(G)$ has a complete and an anti-complete vertex, then

$$\mathrm{index}(G[A]) < \mathrm{index}(G).$$

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If G has twin-width d and index k, then G can be colored with

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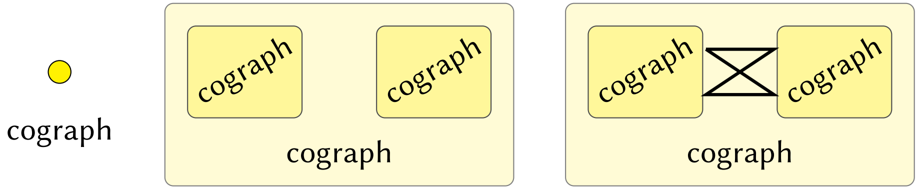

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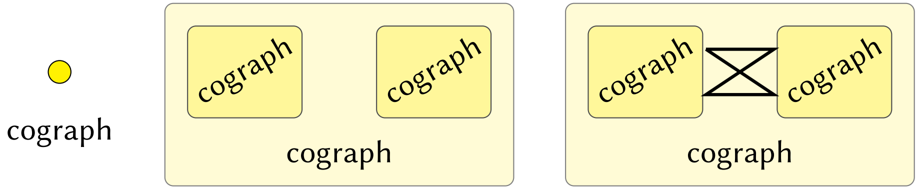

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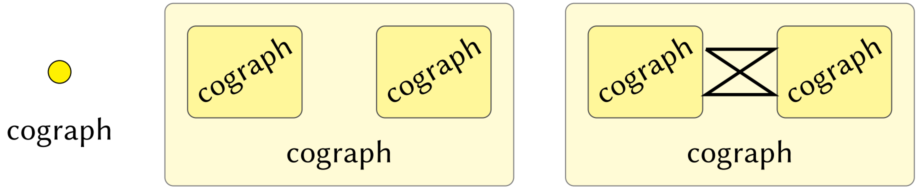

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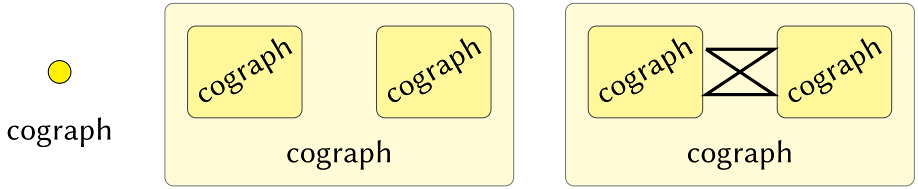

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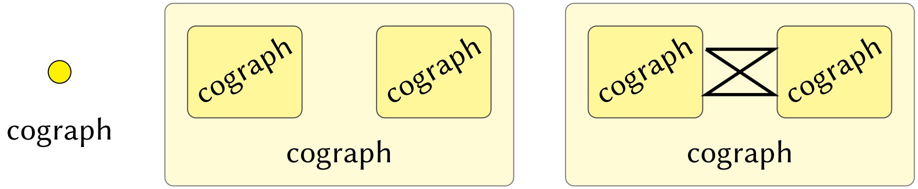

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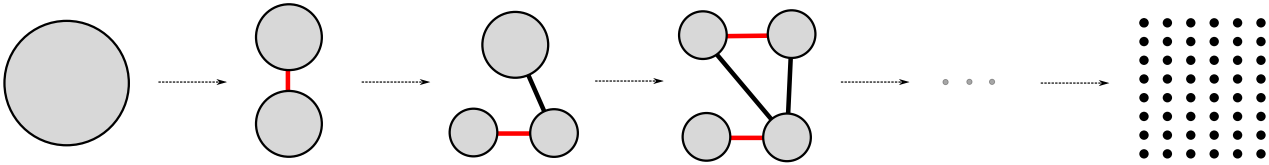

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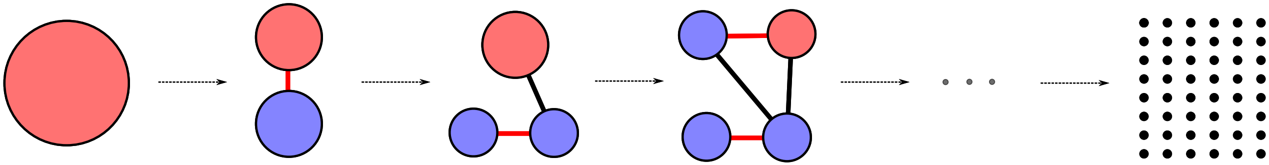

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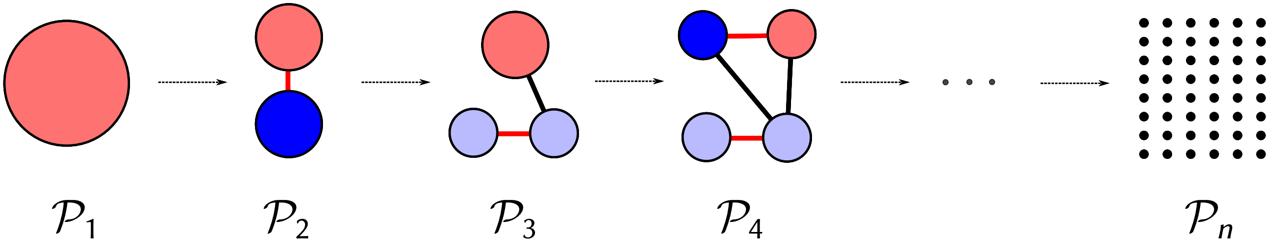

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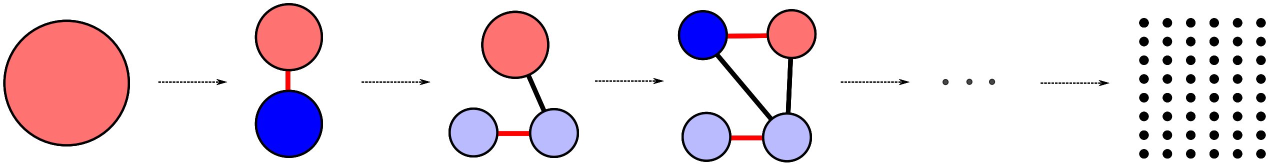

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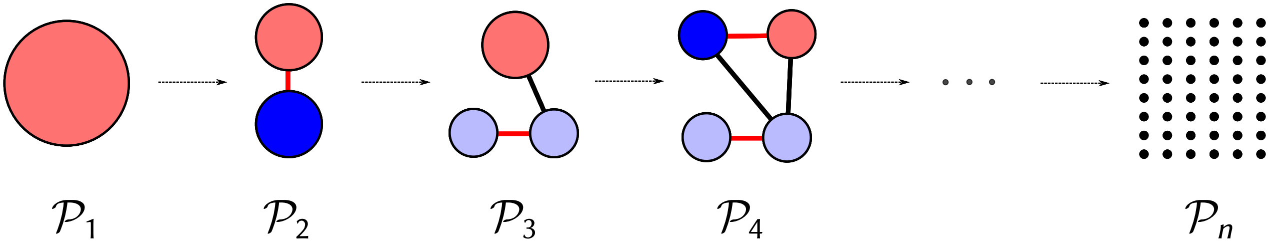

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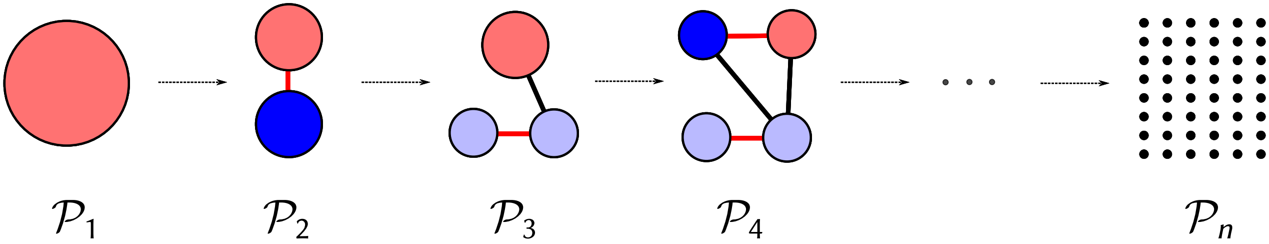

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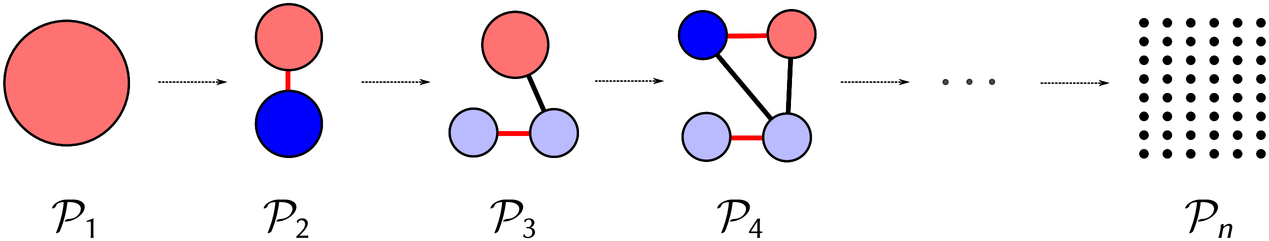

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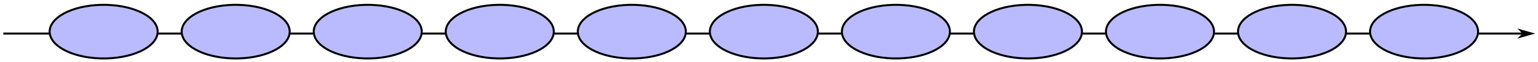

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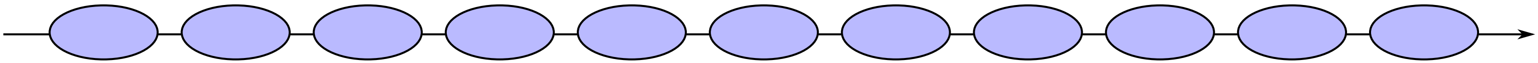

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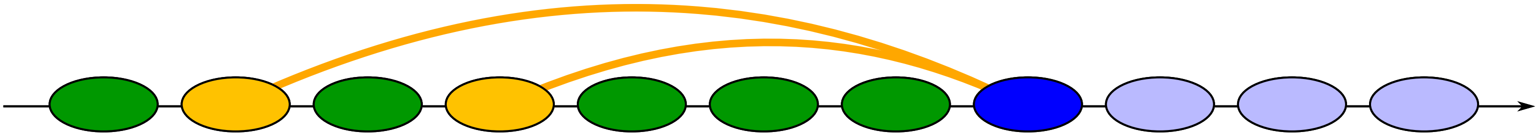

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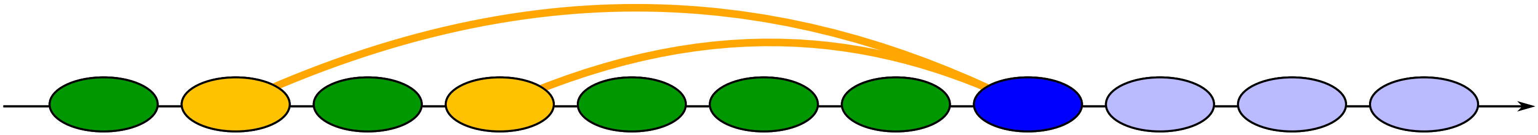

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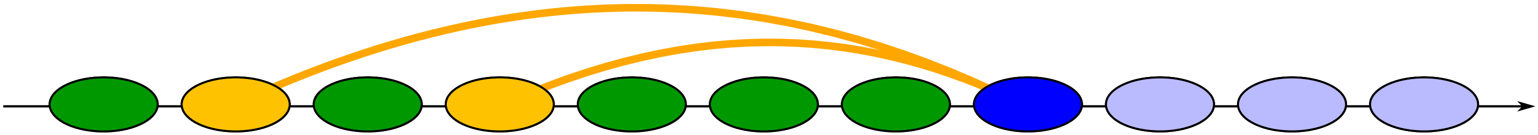

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– $B' :=$ parent of B, say $B' \in \mathcal{P}_t$.

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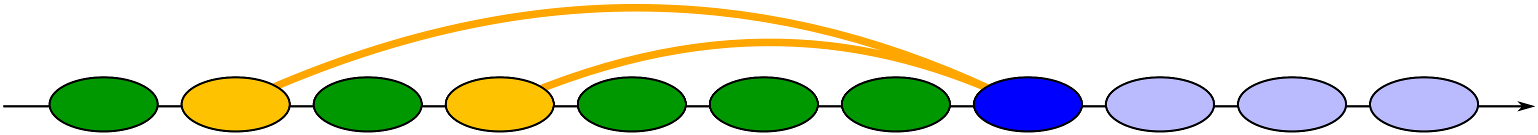

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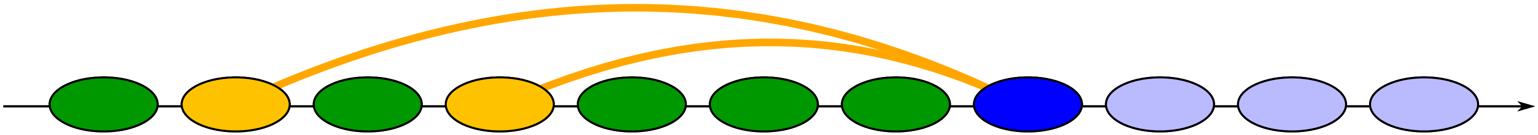

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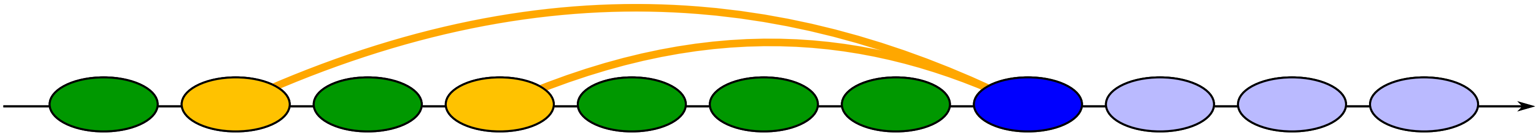

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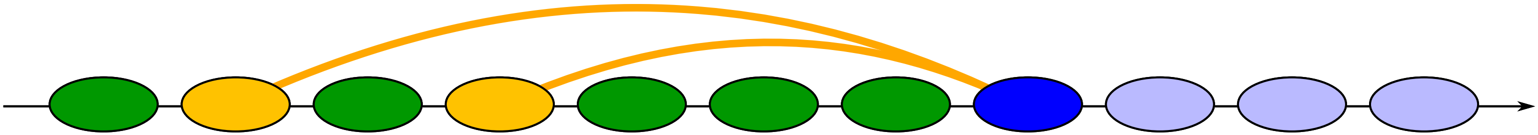

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- $\mathcal{E}_B :=$ frozen ancestors of $\mathcal{N}$ and maybe sibling of B. $\square$

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Partition $\mathcal{F}$ into $2 \cdot (d+2)$ groups:

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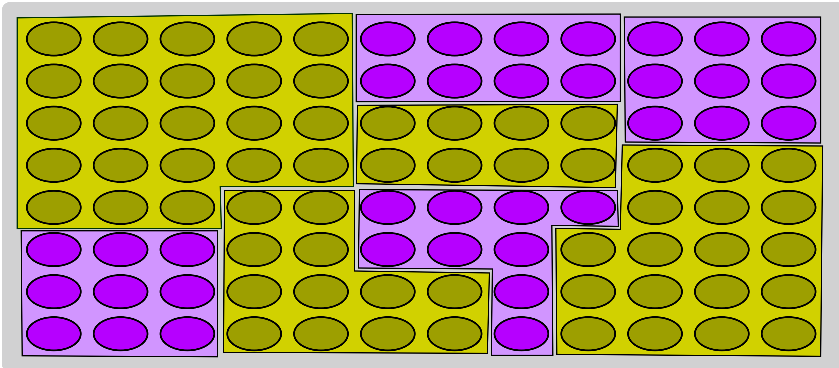

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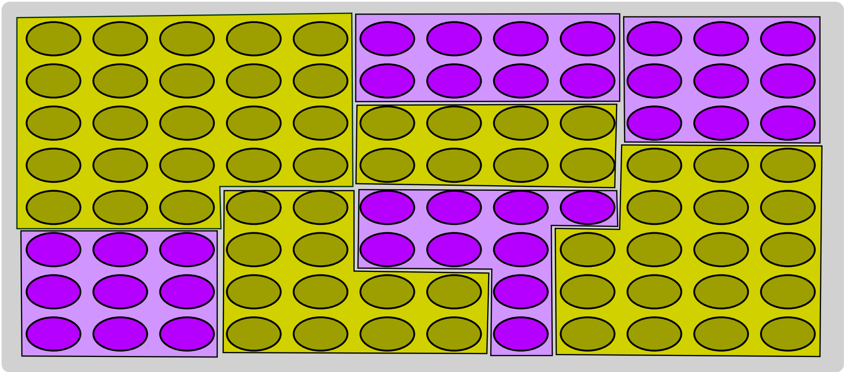

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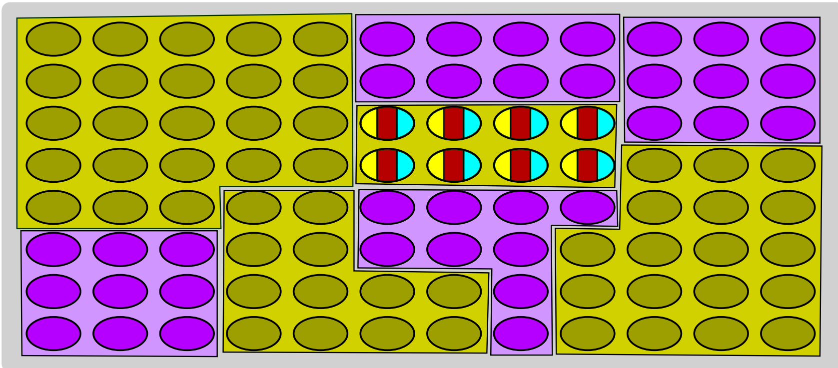

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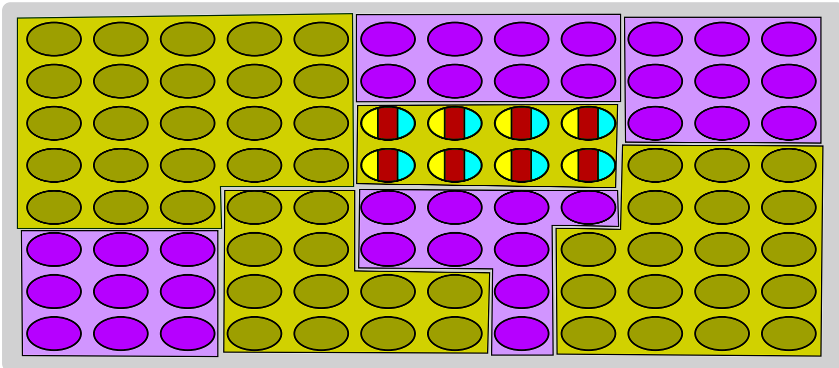

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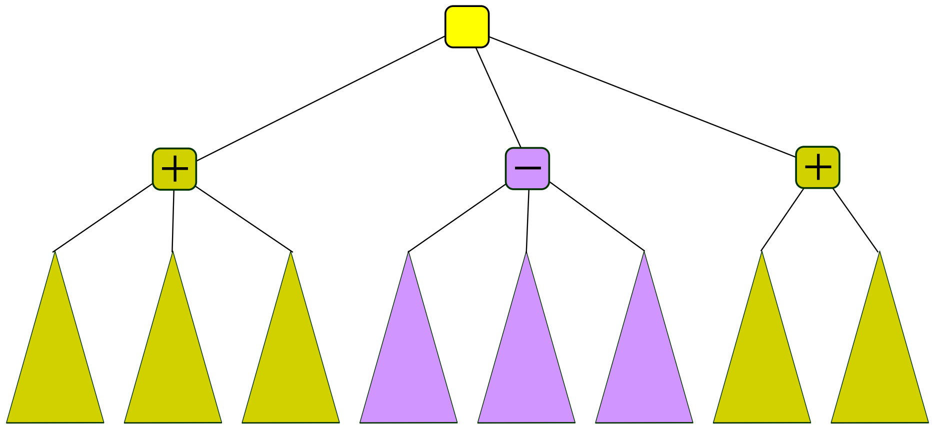

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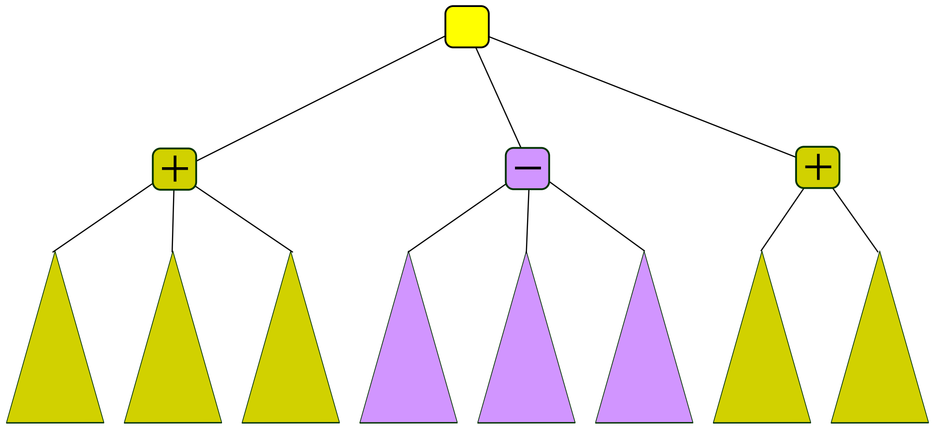

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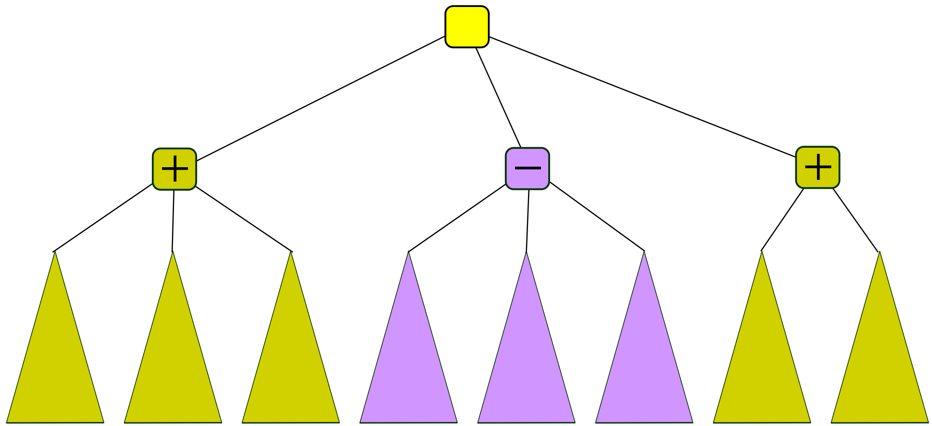

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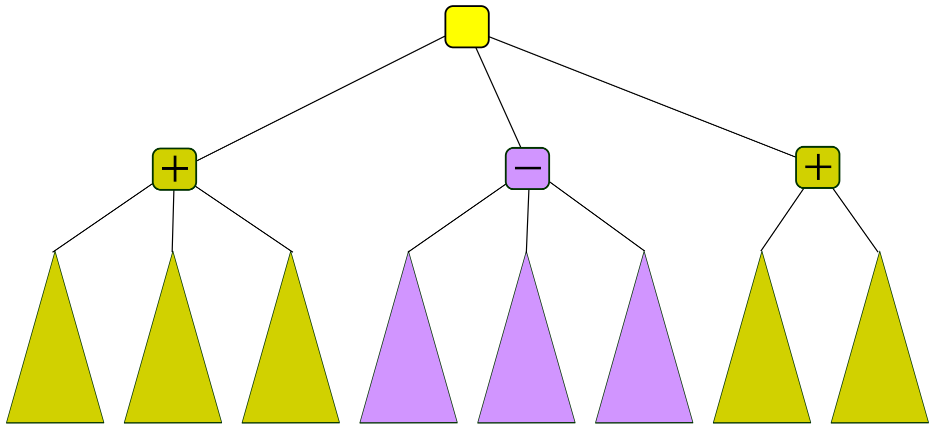

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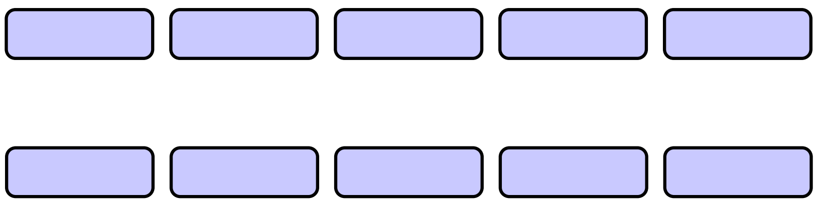

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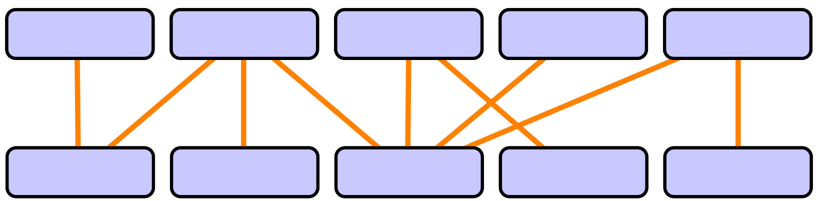

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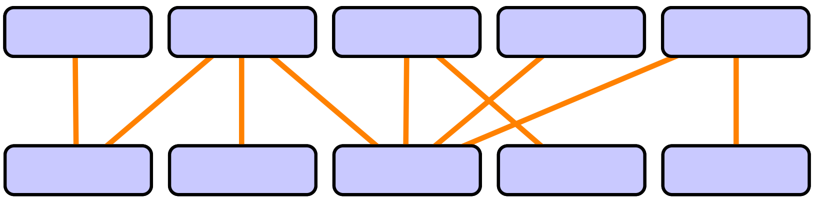

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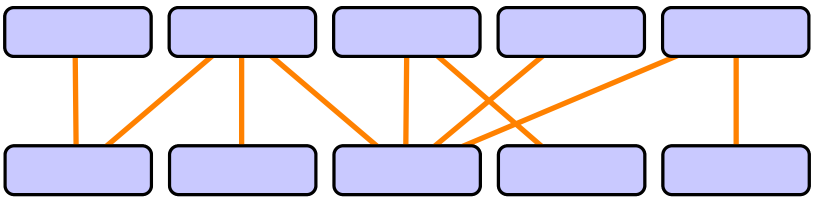

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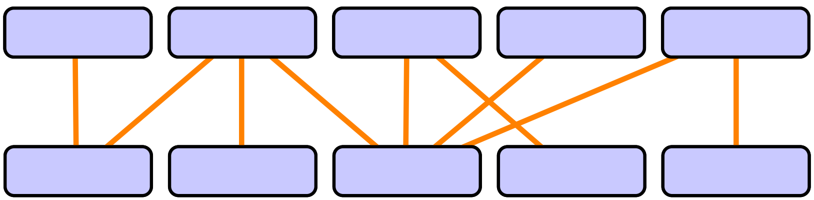

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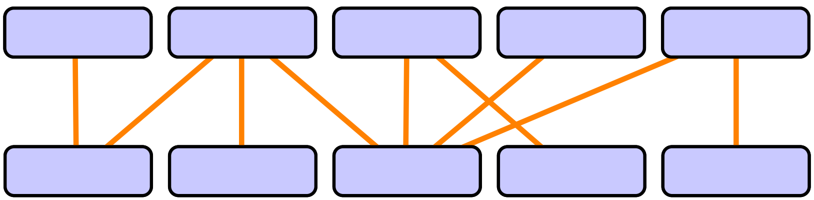

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Theorem (OdMPS'23)

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