Twin-width of Planar Graphs is at most 8

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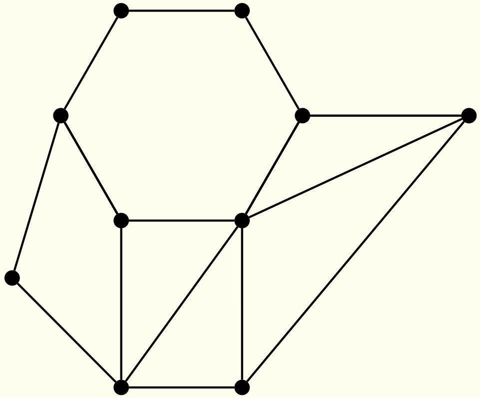

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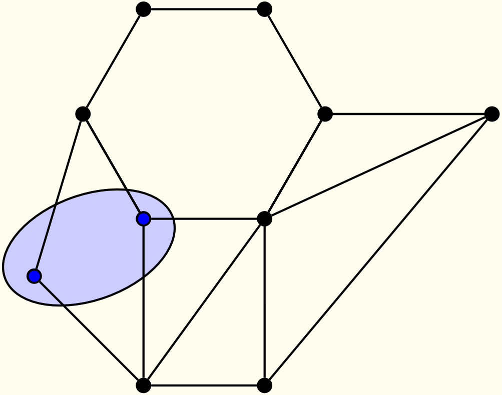

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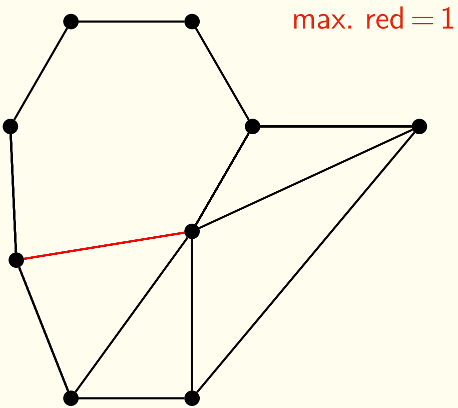

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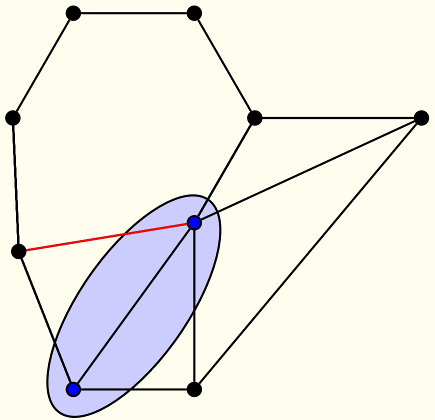

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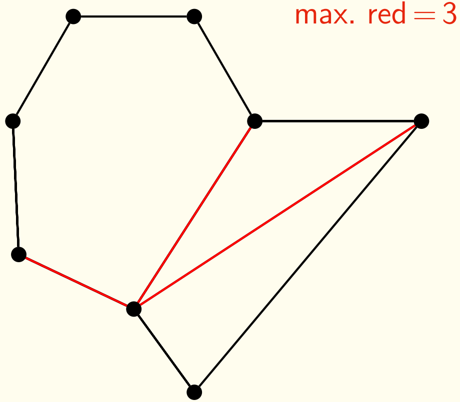

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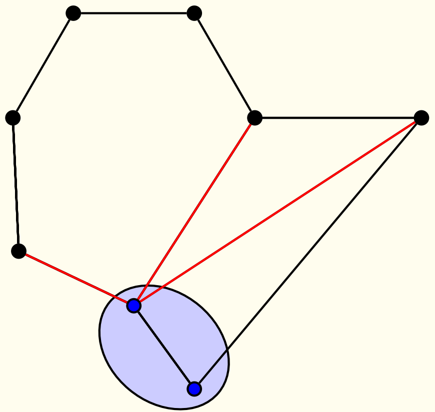

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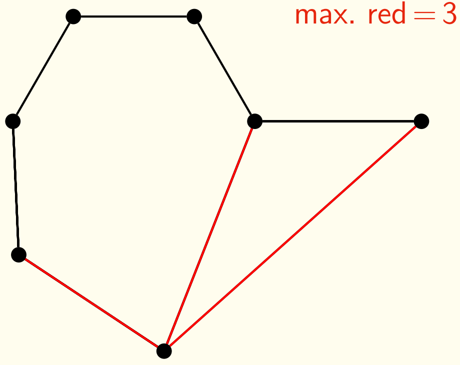

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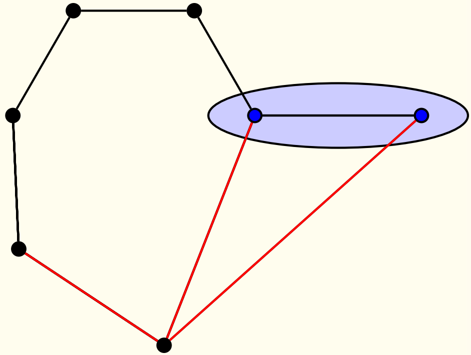

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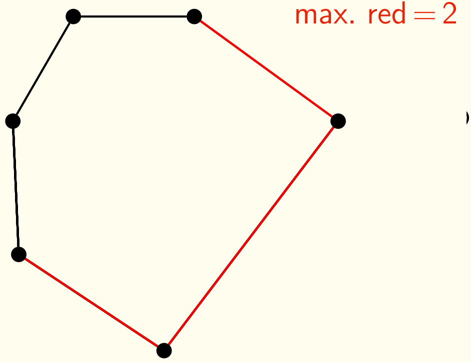

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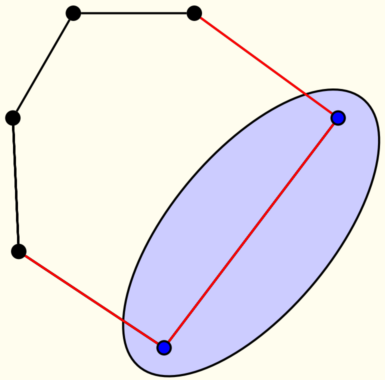

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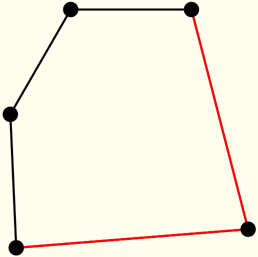

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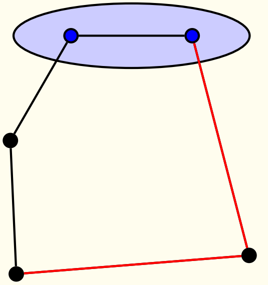

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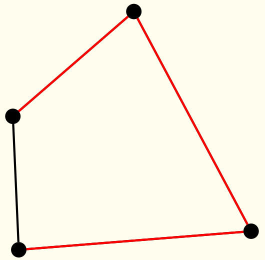

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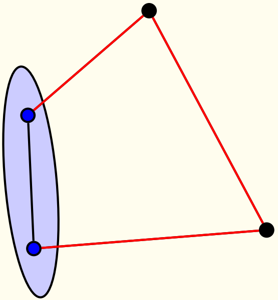

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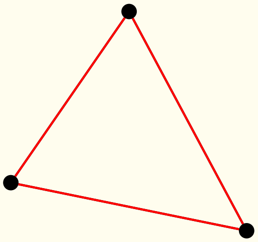

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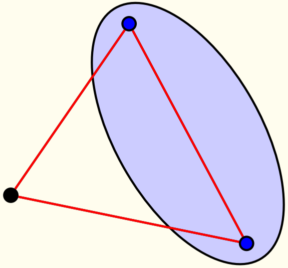

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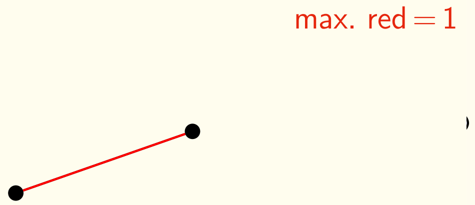

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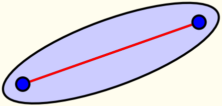

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twin-width ≤ 3

Bounding Twin-Width of Planar Graphs

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Wait a bit longer. . .

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- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

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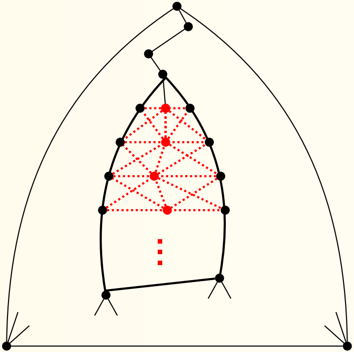

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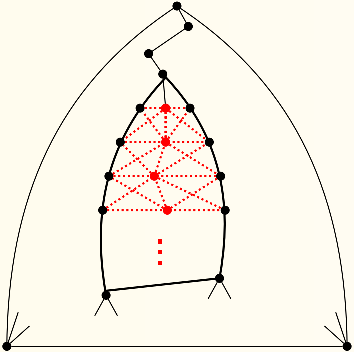

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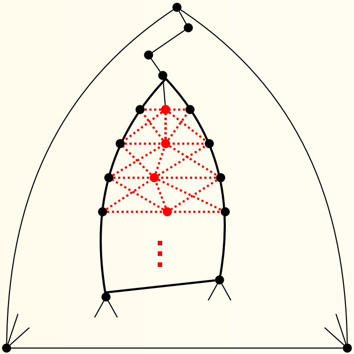

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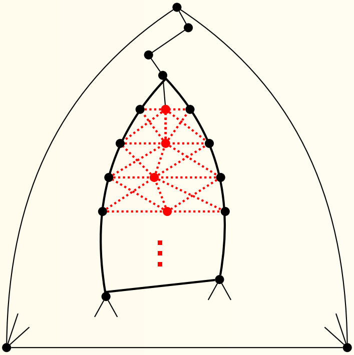

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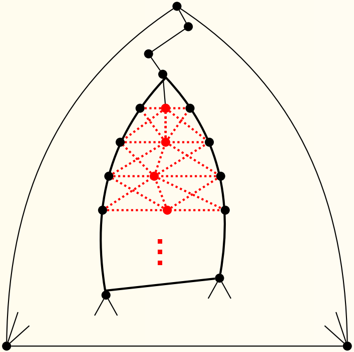

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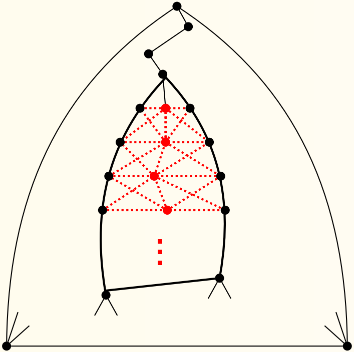

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- Additional detailed conditions for refined proofs. . .

- This setup largely restricts possible red degrees.

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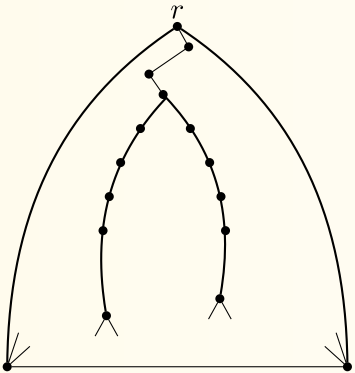
- Choose a root on the outer f., and a BFS tree of G^+ from this root. *Note that all edges are only between same and successive BFS layers.*



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- Proceed by induction…

  merge two skel. faces, contract,
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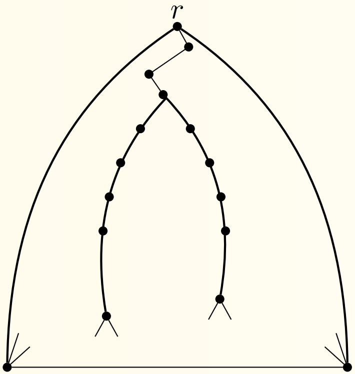
  ……



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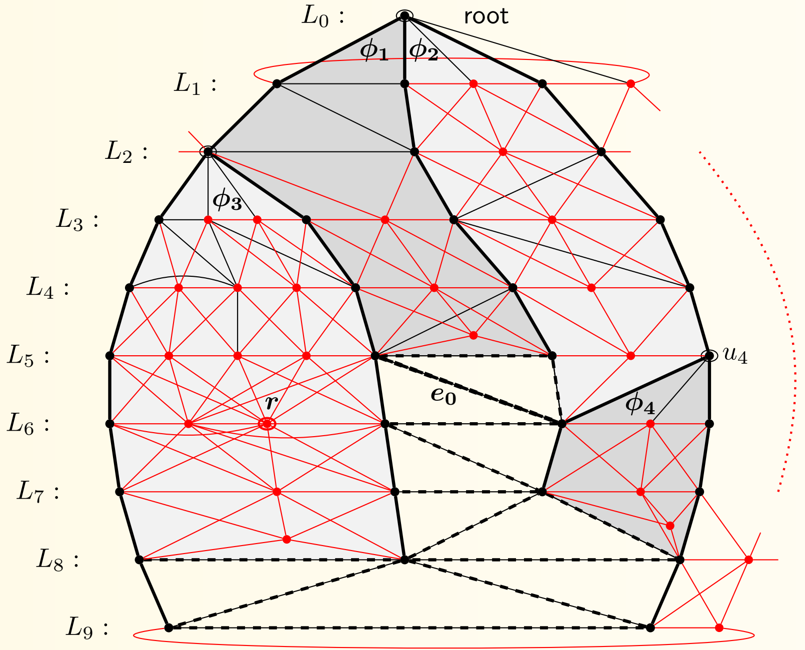
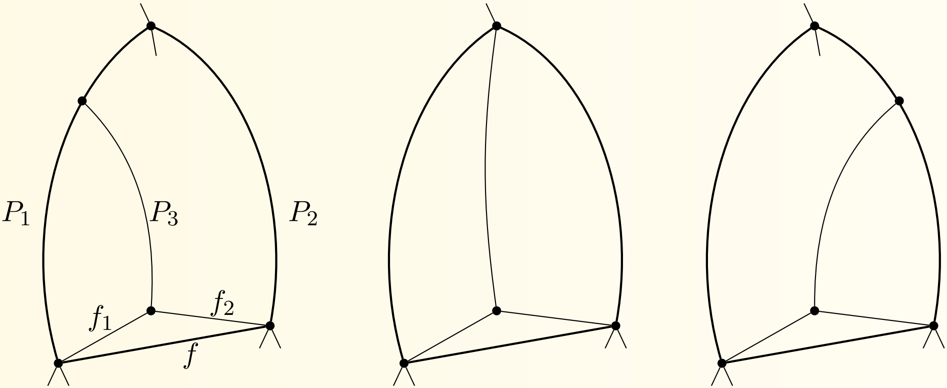


□

- And how do we get to "merge two skeleton faces"?
  - we use a top-down decomposition first...

    (starting from the triangular outer face, and the root at its top)



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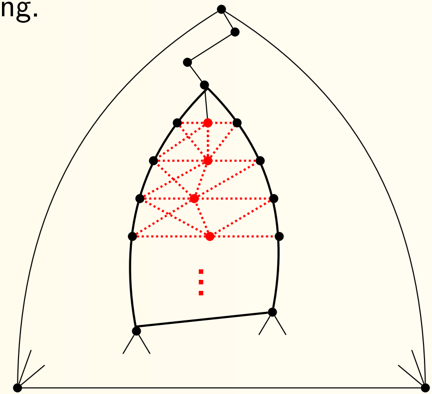
  meaning the shortest paths are chosen more (most) to the left in the drawing.



- Merge skeleton faces across a *vertical-horizontal division* (instead of simply merging two faces)

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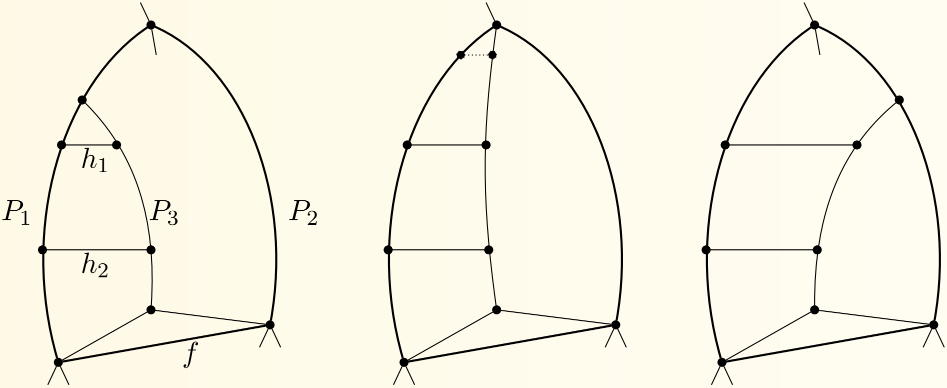

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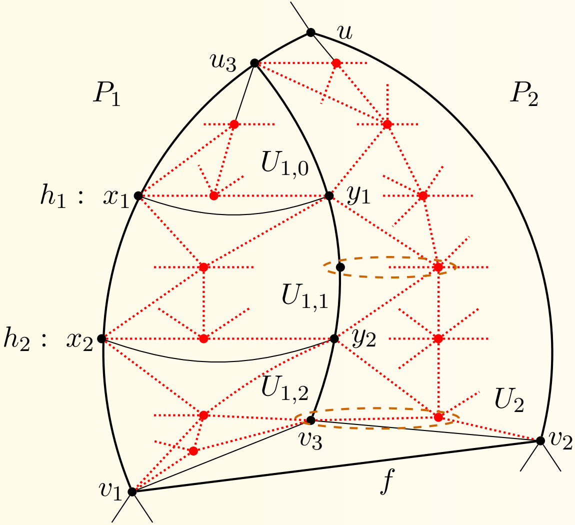
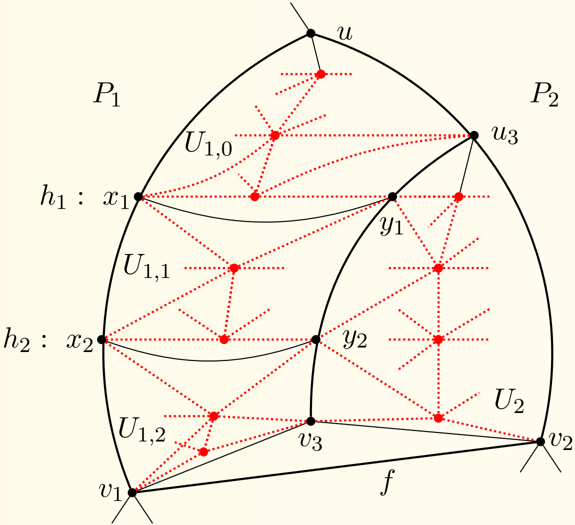
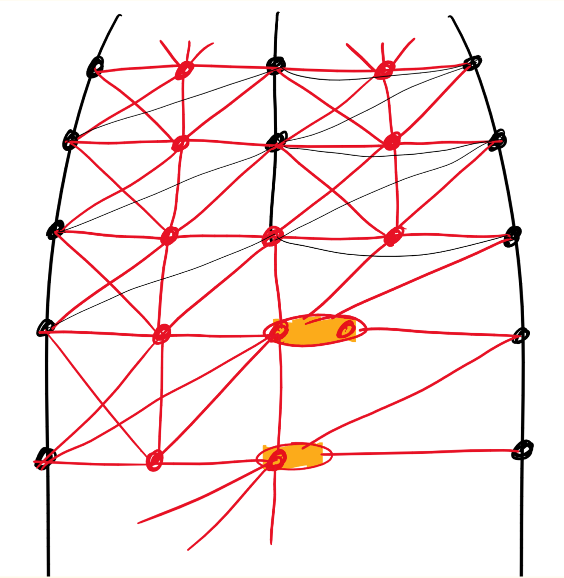
  - bottom-up and right-to-left and bottom-up. . .





u
P_1
P_2
u_3
$U_{1,0}$
$h_1 :\ x_1$
y_1
$U_{1,1}$
$h_2 :\ x_2$
y_2
U_2
$U_{1,2}$
v_3
v_2
f
v_1

- A local detail of the contractions when merging. . .



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Thank you for your attention.