Logic and Twin-width

Sebastian Siebertz

Universität Bremen

1st Workshop on Twin-Width
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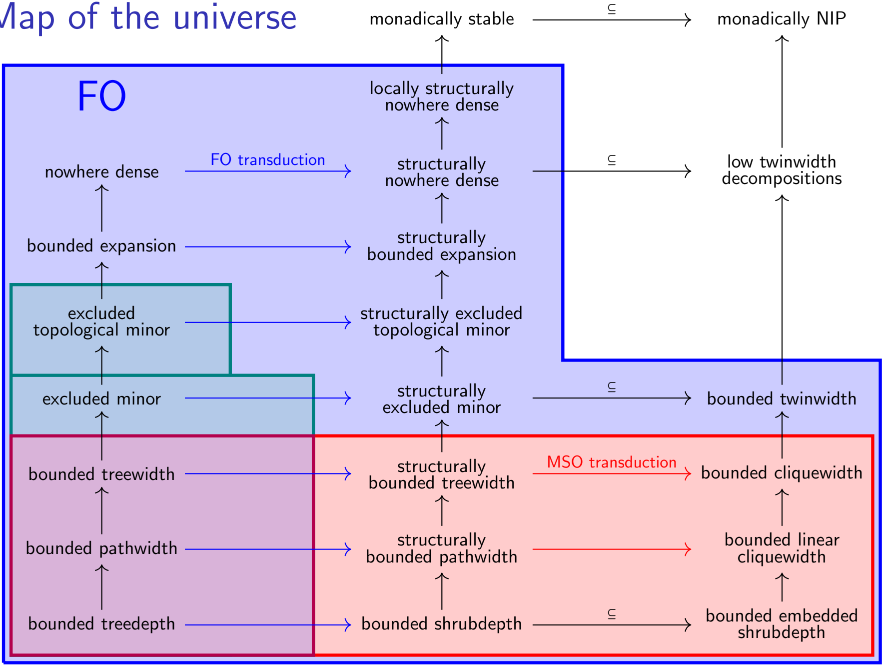
Map of the universe



Stability and dependence

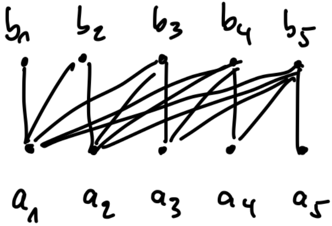
- A graph G is definable in a structure H if there is a formula $\varphi(\bar{x}, \bar{y})$ such that $V(G) = V(H)^{|\bar{x}|}$ and $E(G) = \{(\bar{a}, \bar{b}) \ : \ H \models \varphi(\bar{a}, \bar{b})\}$.
- Example: Finding d-degenerate graphs in the age of edgeless graphs.
  - $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}| = d + 1$
  - Assume $|V(G)| = n$. Let $V(H) = [n] \quad \longrightarrow \quad V(G) \subseteq [n]^{d+1}$.
  - G is d-degenerate $\quad \longrightarrow \quad$ order $V(G)$ such that every $v \in V(G)$ has at most d smaller neighbors, say $v_1, \ldots, v_k$ for $k \leq d$.
  - Map v to $(v_1, \ldots, v_k, v, \ldots, v)$.
  - Define $E(G)$ in H by

$$\varphi(\bar{x}, \bar{y}) = \bigvee_{1 \leq i \leq d} (x_i = y_{d+1} \vee y_i = x_{d+1}).$$

  - Attention: we do not interpret G but a supergraph of G.

Stability and dependence

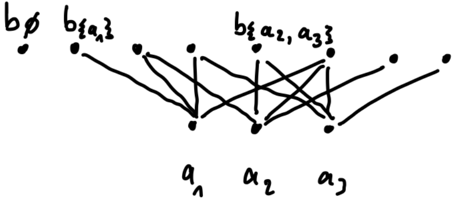
- A class $\mathscr{C}$ of graphs is definable in a class $\mathscr{D}$ of structures if there is a formula $\varphi(\bar{x}, \bar{y})$ such that every $G \in \mathscr{C}$ is defined by φ in some $H \in \mathscr{D}$.
- The order-dimension of a graph G is the largest integer ℓ such that there exist vertices $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell$ with $\{a_i, b_j\} \in E(G) \Leftrightarrow i \leq j$.



- A class $\mathscr{C}$ of structures is stable if every graph class definable in $\mathscr{C}$ has bounded order-dimension.

Stability and dependence

- The VC-dimension of a graph G is the largest integer d such that there exist vertices $a_1, \ldots, a_d \in V(G)$ and vertices $b_J \in V(G)$ for $J \subseteq [d]$ such that $\{a_i, b_J\} \in E(G) \Leftrightarrow i \in J$.



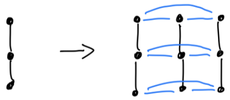
- A class $\mathscr{C}$ of structures is dependent/NIP if every graph class definable in $\mathscr{C}$ has bounded VC-dimension.

Monadic stability and dependence

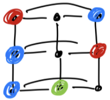
- A class $\mathscr{C}$ of structures is monadically stable/NIP if the class of all monadic expansions (colorings) of structures from $\mathscr{C}$ is stable/NIP.
- Example: The class of 1-subdivided cliques is stable but not monadically NIP.
- [Braunfeld and Laskoswki, 22]: A hereditary class of graphs is stable/NIP if and only if it is monadically stable/NIP.
  - Twin-width is hereditary → we only have to show monadic NIP.
- [Baldwin and Shelah, 85]: $\mathscr{C}$ is monadically stable/NIP if and only every graph class definable in the monadic expansions of $\mathscr{C}$ by formulas $\varphi(x, y)$ have bounded order/VC dimension.
  - Instead of interpretations (in powers) we may look at transductions.
  - Transductions combine colorings and simple interpretations $\varphi(x, y)$.

Transductions

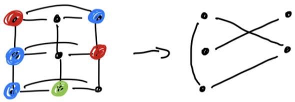
- *k*-copy operation



- Coloring

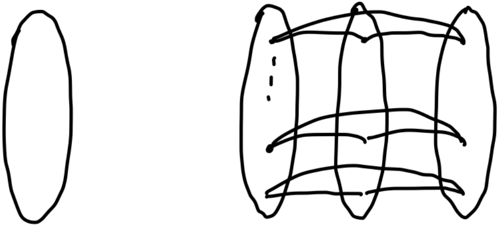


- Simple interpretation $\varphi(x,y)$ defining the new edge set and taking an induced subgraph
  - $\varphi(x,y) = \neg E(x,y)$ (complementing the edge relation)
  - Keep only red and blue vertices (definable induced subgraph)



Bounded twin-width is preserved under transductions

- [Bonnet, Kim, Thomassé, Watrigant, 20]: If $\mathscr{C}$ has bounded twin-width, then every transduction of $\mathscr{C}$ has bounded twin-width.
  - Twin-width is preserved under the k-copy operation:



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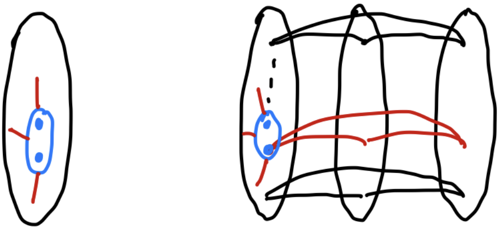

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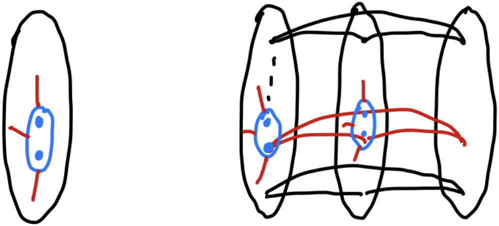

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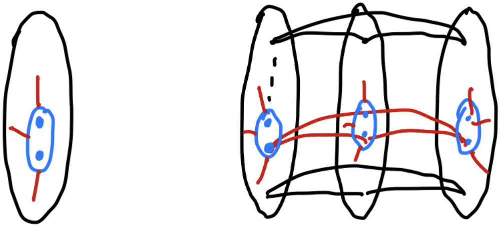

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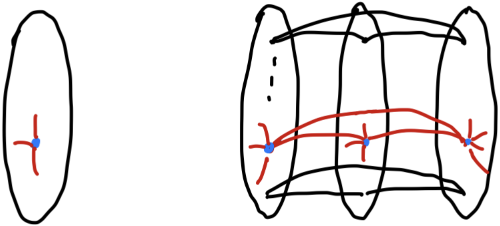


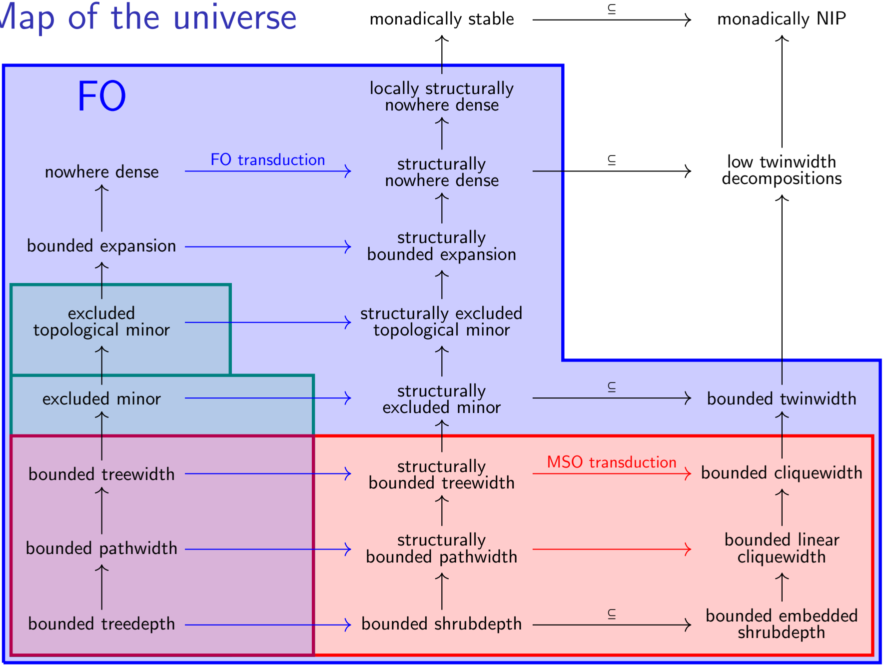
Bounded twin-width is preserved under transductions

- [Bonnet, Kim, Thomassé, Watrigant, 20]: If $\mathscr{C}$ has bounded twin-width, then every transduction of $\mathscr{C}$ has bounded twin-width.
  - Twin-width is preserved under the k-copy operation: ✓
  - Coloring and simple interpretation:
    - Refine the contraction sequence by local red types. [Beautiful presentation by Gajarský, Pilipczuk, Przybyszewski, Toruńczyk, 22]
    - In the contraction sequence local red types change only in local red neighborhoods and can be updated efficiently.

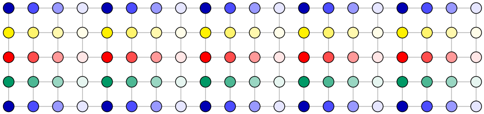
Bounded twin-width classes are monadically NIP

- [Bonnet, Kim, Thomassé, Watrigant, 20]: If $\mathscr{C}$ has bounded twin-width, then every transduction of $\mathscr{C}$ has bounded twin-width.
  - Bounded twin-width classes are monadically NIP.
- [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 22]: A hereditary class of ordered binary structures has bounded twin-width if and only if it is monadically NIP.
- [Bonnet, Kim, Thomassé, Watrigant, 20]: If $\mathscr{C}$ has bounded twin-width and each $G \in \mathscr{C}$ is given with a contraction sequence, then FO model checking is FPT linear on $\mathscr{C}$.
  - Open problem: How to compute good contraction sequences?
- [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 22]: If $\mathscr{C}$ is a hereditary class of ordered graphs, then FO model checking is FPT on $\mathscr{C}$ if and only if $\mathscr{C}$ has bounded twin-width.

Map of the universe



Structural decompositions



- [Nešetřil and Ossona de Mendez, 04]: A class $\mathscr{C}$ of graphs has bounded expansion $\Leftrightarrow$ for every p there exists a class $\mathscr{D}_p$ with bounded treedepth, such that each $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph in $\mathscr{D}_p$.
  - → Classes with bounded expansion have bounded treedepth decompositions.
- [Nešetřil and Ossona de Mendez, 04]: A class $\mathscr{C}$ of graphs is nowhere dense $\Leftrightarrow$ for every p there exists a class $\mathscr{D}_p$ with bounded treedepth, such that each graph $G \in \mathscr{C}$ can be partitioned into at most $N_p \in |G|^{o(1)}$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$.
  - → Nowhere dense classes have quasi-bounded treedepth decompositions.

Structural decompositions

- [Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, S., Toruńczyk, 20]: A class $\mathscr{C}$ of graphs has structurally bounded expansion $\Leftrightarrow$ for every there exists a class $\mathscr{D}_p$ with bounded shrubdepth, such that each $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph in $\mathscr{D}_p$.
  - → Classes with structurally bounded expansion have bounded shrubdepth decompositions.
- [Dreier, Gajarský, Kiefer, Pilipczuk, Toruńczyk, 22]: If a class $\mathscr{C}$ of graphs is structurally nowhere dense, then for every p there exists a class $\mathscr{D}_p$ with bounded shrubdepth, such that each $G \in \mathscr{C}$ can be partitioned into at most $N_p \in |G|^{o(1)}$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$.
  - → Structurally nowhere dense classes have quasi-bounded shrubdepth decompositions.

Structural decompositions

- Let Π be a hereditary class property. If every class $\mathscr{C}$ with quasi-bounded Π decompositions again has property Π, we call Π a decomposition horizon.
- [Braunfeld, Nešetřil, Ossona de Mendez, S., 22] The class properties monadic stability and monadic NIP are decomposition horizons.
  - Classes with quasi-bounded twin-width decompositions are monadically NIP.
- Conjecture: Let $\mathscr{C}$ be a hereditary class of graphs. Then the following are equivalent:
  - $\mathscr{C}$ is structurally nowhere dense
  - $\mathscr{C}$ has quasi-bounded shrubdepth decompositions
  - $\mathscr{C}$ is monadically stable.

Structural decompositions

- Conjecture: Let $\mathscr{C}$ be a hereditary class of graphs. Then
  - $\mathscr{C}$ is monadically NIP

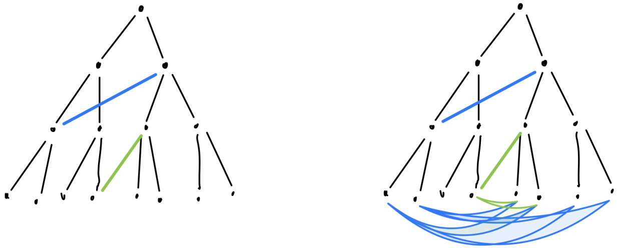
  $\Leftrightarrow$ $\mathscr{C}$ has quasi-bounded twin-width decompositions.
- [Bonnet, Geniet, Kim, Thomassé, Watrigant, 21]: Sparse classes with bounded twin-width have bounded expansion, hence bounded treedepth decompositions.
- [Gajarský, Pilipczuk, Toruńczyk, 22]: Stable classes with bounded twin-width have structurally bounded expansion, hence bounded shrubdepth decompositions.

Structural decompositions

- [Ossona de Mendez]: Classes with bounded twin-width cannot be decomposed into a constant number of simpler pieces (no bounded decompositions):
  - Take a class $\mathscr{C}$ with bounded twin-width and unbounded X (e.g. unbounded cliquewidth).
  - Close $\mathscr{C}$ under lexicographic products (this preserves twin-width).
  - Color with a bounded number of colors.
  - Then we find a monochromatic copy of any $G \in \mathscr{C}$ (Ramsey).
  - Hence, $\mathscr{C}$ does not have bounded X decompositions.

Twin-models

- [Bonnet, Nešetřil, Ossona de Mendez, S, Thomassé, 22]:
  Twin-model: read the contraction sequence the other way around to get a tree-like representation



- Create all tuples (u, v) such that there exists $u' \leq u$ and $v' \leq v$ with a transversal edge (u', v').
  - Need to satisfy a minimality and consistency condition so that we can get a contraction sequence from a twin-model.

Twin-models

- The twin-model together with the tree-order (full twin-model) has bounded twin-width (at most twice the twin-width of the structure).
- The Gaifman graph of the twin-model (without the order) is sparse $\rightarrow$ has bounded expansion.
- Two applications:
  - Construction of sparse weak neighborhood covers.
  - Twin-width and permutations.

Sparse weak neighborhood covers

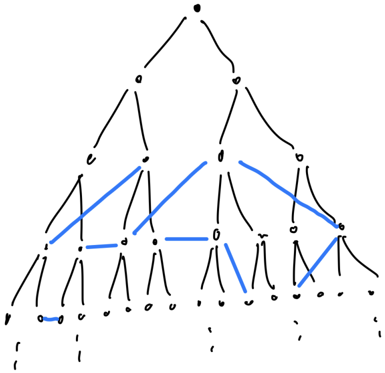
- A weak r-neighborhood cover with degree d and spread s of a graph G is a family $\mathcal{X}$ of subsets of $V(G)$, called clusters, such that
  - the r-neighborhood of every vertex is contained in some cluster: for every $v \in V(G)$ there exists $X \in \mathcal{X}$ with $N_r[v] \subseteq X$,
  - every cluster has weak diameter at most s and
  - every vertex occurs in at most d clusters: for all $v \in V(G)$
$$|\{X \in \mathcal{X} \mid v \in X\}| \leq d.$$
- A class $\mathscr{C}$ admits sparse weak neighborhood covers if there exist functions $g(r, \varepsilon)$ and $s(r) \geq r$ such that for every $r \in \mathbb{N}$, every $\varepsilon > 0$, every graph $G \in \mathscr{C}$ admits a weak r-neighborhood cover with degree $g(r, \varepsilon) \cdot |G|^{\varepsilon}$ and spread $s(r)$.

Sparse weak neighborhood covers

- It suffices to look for 1-neighborhood covers: An r-neighborhood cover can be recovered from a 1-neighborhood cover in the rth power of G → this is a transduction, hence we again have bounded twin-width.
- Let $A \subseteq V(G)$. The contraction of A into a single vertex is a weak k-contraction if A has weak radius at most k, that is, there is $v \in V(G)$ such that $A \subseteq N_k(v)$.
- If H is obtained from G by disjoint weak k-contractions and H admits a weak r-neighborhood cover with degree d and spread s, then G admits a weak r-neighborhood cover with degree d and spread $(2k+1)s$.

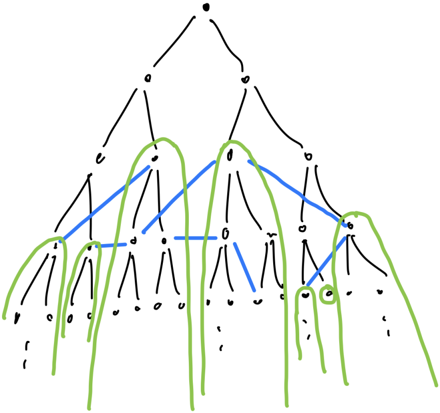
Sparse weak neighborhood covers

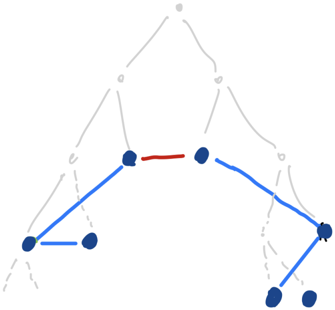
- Consider a full twin-model (with order).



Sparse weak neighborhood covers

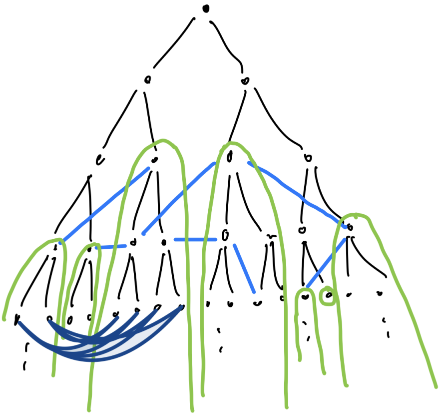
- Consider the minimal elements with traversal edges and contract everything below to single vertices.

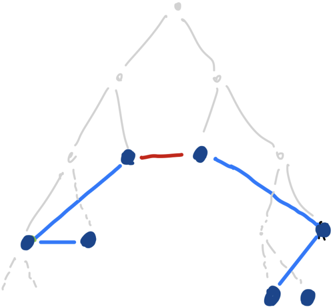




Sparse weak neighborhood covers

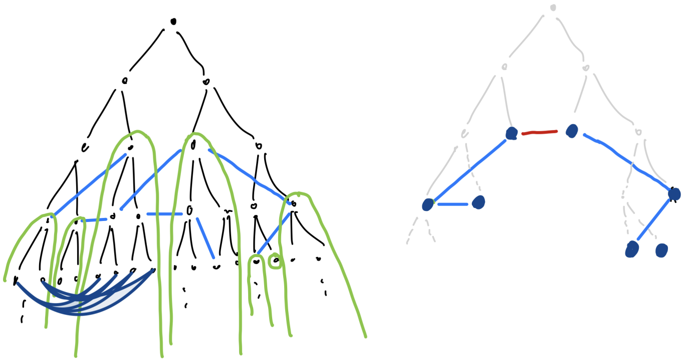
- These are 1-contractions, because the traversal edges encode bicliques.





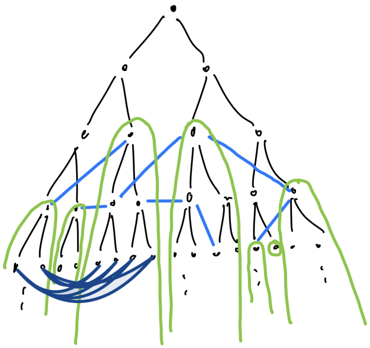
Sparse weak neighborhood covers

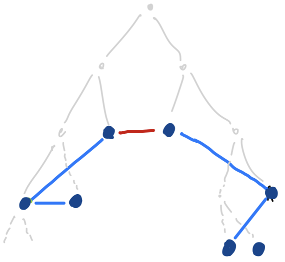
- The contractions are definable in the full twin-model, which has bounded twin-width.
- Hence, the resulting graph has bounded twin-width.



Sparse weak neighborhood covers

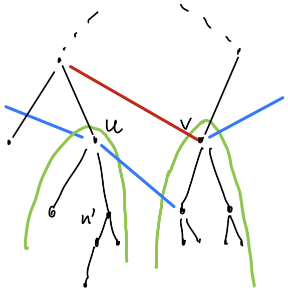
- The resulting edges are either blue (transversal edges that were present in the twin-model).
- These edges alone induce a graph of bounded expansion, because they are a subset of the edges of a graph with sparse bounded twin-width.





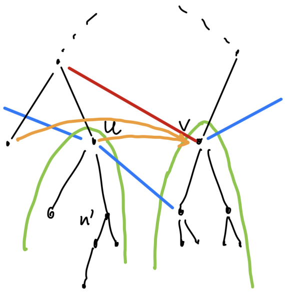
Sparse weak neighborhood covers

- All other edges are created by the contraction.
- Problem: these are not red edges from the contraction sequence.
  - Assume v comes alive first and is no longer alive when u comes alive (reading the sequence backwards).



  - Then a predecessor of u became alive before v and we have a red edge (otherwise there would be a blue edge in the twin-model and u would not be a vertex of the reduced graph).

Sparse weak neighborhood covers



- We push down the red edge(s) (may be many copies) and orient them towards v
- The out-degree remains bounded → we still have a degenerate twin-model (without the order) and hence bounded expansion.
- Classes with bounded expansion have sparse neighborhood covers ✓

Structurally nowhere dense graphs

- [Dreier, Gajarský, Kiefer, Pilipczuk, Toruńczyk, 22]: If a class $\mathscr{C}$ of graphs is structurally nowhere dense, then its graphs have similar tree-like decompositions called quasi-bushes with quasi-bounded weak coloring numbers.
- [Dreier, Mählmann, S., 23]: Structurally nowhere dense graph classes have sparse weak neighborhood covers.
  - Proof: conceptually similar but more technical.
  - Question: Do structurally nowhere dense graph classes have nowhere dense quasi-bushes?
- [Dreier, Mählmann, S., 23]: FO model checking on (locally) structurally nowhere dense classes is fixed-parameter tractable.
  - Flipper game
  - Local types (avoid complicated rank preserving locality)
  - Sparse weak neighborhood covers (can be efficiently approximated)

Model checking on monadically stable classes

- If a monadically stable class $\mathscr{C}$ admits sparse weak neighborhood covers, then FO model-checking is fixed-parameter tractable on $\mathscr{C}$.
- Conjecture: Monadically NIP classes admit sparse weak neighborhood covers.
- Question: Do monadically stable classes have treelike decompositions of bounded depth?

Permutations

- [Bonnet, Nešetřil Ossona de Mendez, S, Thomassé, 2022]: A class of binary relational structures has bounded twin-width if and only if it is a first-order transduction of a proper permutation class.
  - Permutation: two linear orders on the universe $(V, <_1, <_2)$.
  - Proper permutation class: set of permutations closed under sub-permutations excluding at least one permutation.
- Example: 21-avoiding permutation = linear order. Transductions have bounded linear cliquewidth.
- Example: 231-avoiding permutation = tree order. Transductions have bounded cliquewidth.

"$\Leftarrow$" Proper permutation classes have bounded twin-width (small ordered hereditary classes) and so have their transductions.

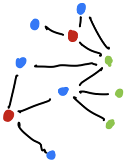
Tree-models and permutations

"$\Rightarrow$" Let $\mathscr{C}$ be a class of bounded twin-width.

- We show that the class $\mathscr{F}$ of full twin-models of graphs from $\mathscr{C}$ is bi-transducible with a permutation class $\mathscr{P}$.
  - We can transduce $\mathscr{C}$ from $\mathscr{F}$.
  - Hence $\mathscr{C}$ is a transduction of the permutation class $\mathscr{P}$.
  - $\mathscr{P}$ is a proper permutation class because it is a transuction of $\mathscr{F}$, which has bounded twin-width, and hence has bounded twin-width.
- The full twin-models of graphs from $\mathscr{C}$ have bounded twin-width and without the order they are sparse (have bounded expansion and in particular have bounded star chromatic number).

Transductions and star chromatic numer

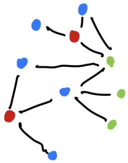
- Star coloring: Proper coloring such that any two color classes induce a star forest (disjoint union of stars).



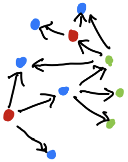
- [Courcelle?]: Let Σ be a relational signature (of arbitrary arity) and $\mathscr{C}$ a class of Σ-structures. Assume the class of Gaifman graphs $\mathscr{G}$ of $\mathscr{C}$ has bounded star chromatic number. Then $\mathscr{C}$ is bi-transducible with $\mathscr{G}$.
  - In particular, we can transduce all orientations of graphs with bounded star chromatic number.

Star chromatic number and transductions

- Let $\mathscr{G}^<$ be a class of ordered graphs with star chromatic number at most c. Then $\mathscr{G}^<$ is bi-transducible with a class $\mathscr{P}$ of permutations.
  - Take a star coloring of G with c colors.

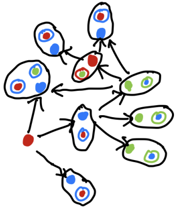


  - Orient the edges so that bicolored stars are oriented away from their centers (every edge is bicolored because we have a proper coloring).



Star chromatic number and transductions

- Blow each vertex into $(u, 1), \dots, (u, c+1)$ and
- keep only the vertices of the form $(u, c+1)$ and (u, i) if u has an in-neighbor colored i.



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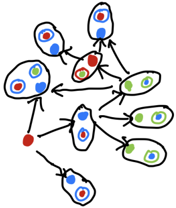

- Define two orders:

Star chromatic number and transductions

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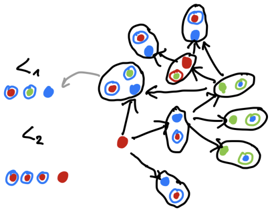
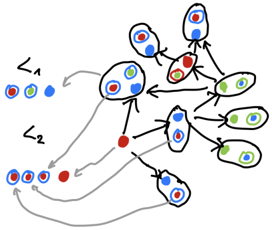


- Define two orders:
  - $<_1$ helps to identify copies – it orders $(u,1),(u,2),\ldots,(u,c+1)$ consecutively.

Star chromatic number and transductions

- Blow each vertex into $(u, 1), \ldots, (u, c+1)$ and
- keep only the vertices of the form $(u, c+1)$ and (u, i) if u has an in-neighbor colored i.



- Define two orders:
  - $<_1$ helps to identify copies – it orders $(u, 1), (u, 2), \ldots, (u, c+1)$ consecutively.
  - $<_2$ helps to recover the edges – it puts a copy (v, i) for an out-neighbor v of u directly in front of $(u, c+1)$.

Map of the universe