Introduction to Twin-Width

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- TU-matrices, perfect graphs, minor closed classes, bounded expansion, pattern-free permutations ...

Complexity of input (static) vs computation (dynamic)

Some features of simple discrete structures

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Often boils down to "Strict vs Full" class (minor closed, pattern-free, bounded VC-dimension)

Counting (Strict vs Full): VC-dimension

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Where are the others gaps?

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Exponential growth is called *small*

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(Nearly) everything in this talk based on MT

Counting (Strict vs Full): Parity minors

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bounded tww $\equiv$ parity minor closure is strict

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(Approximate) counting follows from partitions

Partitions: Szemerédi lemma

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There is a sequence of partitions approximating G

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G approximated by a sequence G/P with few errors

Partitions: Error degree of a partition

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Degree of P is maximum red degree in G/P

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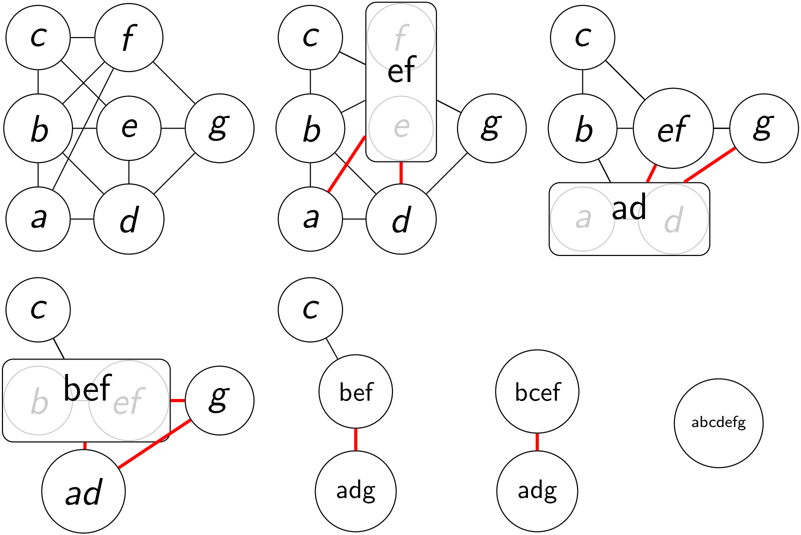
The *twin-width* of G is the minimum degree of a partition sequence S

Partitions: A degree 2 sequence



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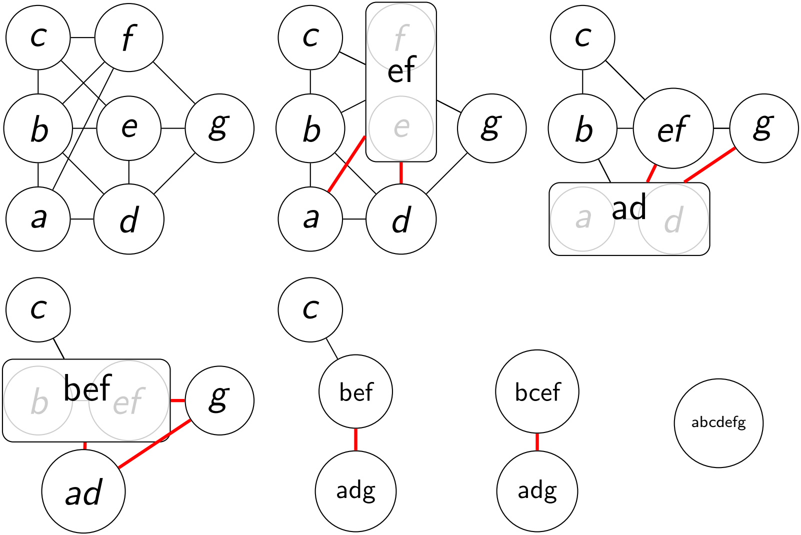

Can we restrict more?

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Twin-width sits between rank-width and bounded VC-dimension

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Partitions are obtained from matrix divisions

Matrix divisions: The Füredi-Hajnal conjecture

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\hline
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Every $n \times n$ matrix with $c_k.n$ "1" have a k-grid minor

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Sparse G: bounded tww $\approx A_G$ has no large grid minor

Matrix divisions: The dense case, mixed-minors

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Pilipczuk and Sokołowski: forget the diagonal

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Grid rank definition works for infinite fields

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How fast can we find an odd set in a planar graph?

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FO+MOD-transduce a total order?

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What are bounded tww polyhedra? Bipartite matching??

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Can we construct H-free graphs? Erdős-Hajnal??