Flip-width

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1. Motivation

extend graph width parameters
from sparse graphs to **dense graphs**

properties of graph classes defined by the parameters

e.g. "bounded treewidth"
(sparse)

and "bounded cliquewidth" = "bounded rankwidth"
(**dense**)

properties of graph classes that exhibit good logical behavior

- Monadic Second Order Logic
  is well behaved on classes of
  bounded treewidth, or bounded cliquewidth.

e.g. model checking is fixed-parameter tractable.

- First order logic
  is well behaved on nowhere dense classes

$$\begin{tikzcd}
\text{sparse} & \textbf{dense} & \\
\text{bounded treedepth} \arrow[r] \arrow[d] & \text{bounded shrubdepth} \arrow[d] & \\
\text{bounded treewidth} \arrow[r] \arrow[d] & \text{bounded cliquewidth} \arrow[d] & \text{vertex-minor free ?} \arrow[l] \\
\text{forbidding a minor} \arrow[r] \arrow[d] & ? \arrow[d] & \\
\text{bounded sparse twin-width} \arrow[r] \arrow[d] & \text{bounded twin-width} \arrow[d] & \\
\text{bounded expansion} \arrow[r] \arrow[d] & ? \arrow[d] & \\
\text{nowhere dense} \arrow[r] & ? &
\end{tikzcd}$$

Unify sparsity & twin-width

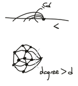
2. Dualities

$$\begin{tikzcd}
& \text{graph parameters / classes} \arrow[dl] \arrow[dr] & \\
\text{decompositions} \atop \sim\text{width} \arrow[rr, leftrightarrow, "\text{duality}"] & & \text{obstructions}
\end{tikzcd}$$

Examples

- (Kuratowski)
  G is planar $\Leftrightarrow$ no $K_{3,3}$ or K_5-minor
- (Robertson & Seymour)
  $\mathcal{C}$ has bounded treewidth
  $\Leftrightarrow$ $\mathcal{C}$ avoids some grid as a minor.
- (Seymour & Thomas)
  G has treewidth $\leqslant k$
  $\Updownarrow$
  G has no bramble of order k
- Grid theorem for rankwidth (Geelen-Kwon-McCarty-Wollan '23)
  $\mathcal{C}$ has bounded rankwidth
  $\Leftrightarrow$ $\mathcal{C}$ avoids some circle graph as a vertex minor.
- (Oum-Seymour '2006)
  G has rankwidth $\leqslant k$
  $\Updownarrow$
  G has no well linked set of size $> \Theta(k)$

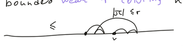
- G has degeneracy $\leq d$
  $\Leftrightarrow$ G does not contain a subgraph with minimum degree $>d$



- Bounded expansion (Nešetřil, Ossona de Mendez, Zhu)
  Fix $1 \leq r < \infty$.

either

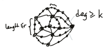
$\mathcal{C}$ has bounded weak r-coloring numbers



$\forall G \in \mathcal{C}. \exists \leq. \forall v. \left|\{\min_{\leq}(\pi) : \pi\text{-path from } v, |\pi| \leq r\}\right| = O(1)$

or

$\mathcal{C}$ contains a $\leq(r-1)$-subdivision of a graph with min.deg. $\geq k$, for all $k \in \mathbb{N}$



$\mathrm{wcol}_r(\mathcal{C}) < \infty$: $\exists k \in \mathbb{N}. \forall G \in \mathcal{C}\ \exists <$ on V:

$\forall v \in V(G). \exists^{\leq k} w \in V(G). \exists\ v\text{-}w$ path π: $|\pi| \leq r$ & $\min(\pi) = w$.

$\mathrm{wcol}_r(G)$ — $\min\{k$: G has an ordering s.t. $\forall v \in V(G)$, $\leq k$ vertices w can be reached from v by a path of length $\leq r$ with $\min \pi = w\}$

$\overset{?}{\approx}$

$\widetilde{\nabla}_r(G)$ — $\max\{\mathrm{mindeg}(H)$: H-graph whose $\leq r$-subdivision H' is a subgraph of G$\}$

- Twin-width

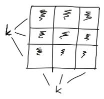
A class $\mathcal{C}$ of ordered graphs $G = (V, E, <)$

either $\mathcal{C}$ has bounded twin-width

$\Updownarrow$

or $\mathcal{C}$ has k-rich-divisions, for all $k \in \mathbb{N}$

Adj(G)



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\text{bounded sparse twin-width} \arrow[r] \arrow[d] & \text{bounded twin-width} \arrow[d] \\
\text{bounded expansion} \arrow[r] \arrow[d] & ? \arrow[d] \\
\text{nowhere dense} \arrow[r] & ?
\end{tikzcd}$$

Parameters via games

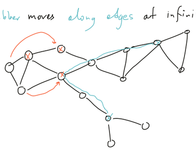
Treewidth : Cops & Robber (Seymour & Thomas)

k – fixed parameter

- k Cops in helicopters,
- one robber.

In each round:

- Cops announce new positions & fly
- Robber moves along edges at infinite speed



3. Copwidth

SPEED
LIMIT
25

$r \in \mathbb{N} \cup \{\infty\}$ range / speed of robber

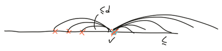
$\mathrm{copwidth}_r(G) :=$ minimum number k such that k cops can catch a robber of speed r

r	$\mathrm{copwidth}_r$
$r = \infty$	treewidth + 1
$r = 1$	degeneracy + 1
$1 \leq r < \infty$	$\sim \Theta(r)$-weak-coloring number

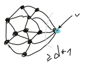
$\boxed{r = 1}$

$\mathrm{copwidth}_1(G) = \mathrm{degeneracy}(G) + 1$

G has degeneracy d
$\Downarrow$
$d+1$ cops suffice



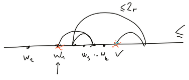
G has degeneracy $> d$
$\Downarrow$
$d+1$ cops do not suffice



$\boxed{1 \leq r < \infty}$

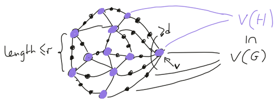
$$\begin{gathered}
2r\text{-weak coloring number}(G) \\
\geq \\
\mathrm{copwidth}_r(G) \\
\geq \\
\max\{k : G \text{ contains an } \leq(r-1)\text{-subdivided graph of mindeg } k\}
\end{gathered}$$

$2r$-weak coloring number$(G) \leq k$
$\Downarrow$
k cops suffice



place cops on vertices w
that are 2r-weakly reachable from v.

G contains some ≤(r-1)-subdivision of
a graph with min. degree ≥ d



d cops do not suffice:

Robber uses V(H) as a hideout

Moral

$$\begin{tikzcd}
\text{decompositions} \arrow[r] \arrow[d, leftrightarrow] & \text{cop strategies} \arrow[d, leftrightarrow] \\
\text{obstructions} \arrow[r] & \text{hideouts for robber}
\end{tikzcd}$$

1.

2. Define parameters via games

3. $\mathcal{C}$ has bounded expansion

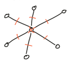
$\Updownarrow$

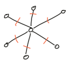
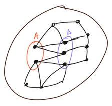
$\text{copwidth}_r(\mathcal{C}) < \infty$, for all $r \in \mathbb{N}$.

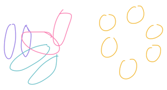
4. Flip-width

Cops & robber on dense graphs

Sparse game: cops block vertices



Sparse	Dense
isolating a vertex	flipping a pair A,B ⊆ V
	



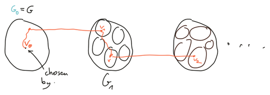
Fix $r \in \mathbb{N} \cup \{\infty\}$.

Sparse	Dense
radius-r Cops & robbers game	radius-r Flipper game
cops isolate k vertices in G robber runs at speed r	Flipper does k-flips runner runs at speed r

Def:

Flipper game of width $k \in \mathbb{N}$ and radius $r \in \mathbb{N} \cup \{\infty\}$ on G:

$G_0 := G$, $v_0 \in V(G)$ chosen by Runner



round $i = 1, 2, \ldots$ G_i - k-flip of G chosen by Flipper
$v_i \in B^r_{G_{i-1}}(v_{i-1})$ - chosen by Runner

Flipper wins when v_k is isolated in G_k.

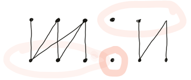
Def. Radius-r flip-width ($r \in \mathbb{N} \cup \{\infty\}$):

$$fw_r(G) := \min\{k \in \mathbb{N} : \text{Flipper wins the game of width } k \text{ and radius } r\}$$

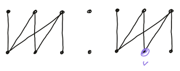
$\boxed{r = \infty}$

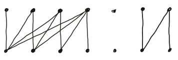
Examples:

- Half graphs.



$fw_\infty(G) \leq 4$





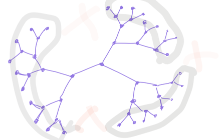
$(r = \infty)$

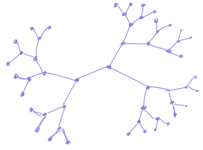
- Graphs of bounded cliquewidth/rankwidth.

$\text{rankwidth}(G) \leq 3\, fw_\infty(G) + 1 \leq O(2^{\text{rankwidth}(G)})$.

Upper bound: $fw_\infty(G) + 1 \leq O(2^{\text{rankwidth}(G)})$.

$k := \text{rankwidth}(G)$





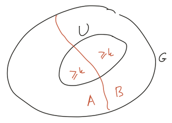
$\exists\, T$ – tree:

- $\text{Leaves}(T) = V(G)$
- Every edge e of T induces a partition $V(G) = A \,\dot\cup\, B$ of rank $\leq k$.

Lower bound: $\text{rankwidth}(G) \leq 3\, fw_\infty(G) + 1$
(by Rose McCarty)

Suppose $\text{rankwidth}(G) > 3k+1$

Then G contains a well-linked set U with $|U| > 3k$



In the radius-∞ game of width k, runner can hide in U indefinetely.

$\boxed{1 \leq r < \infty}$

Definition A graph class $\mathcal{C}$ has bounded flip-width
$\Updownarrow$
$fw_r(\mathcal{C}) < \infty$ for all $1 \leq r < \infty$

$$\begin{tikzcd}
\text{sparse} & & \text{dense} \\
\text{bounded treewidth} \arrow[rr, "r=\infty"] \arrow[d, phantom, "="] & & \text{bounded cliquewidth} \arrow[d, phantom, "="] \\
\text{bounded copwidth}_\infty \arrow[d] & & \text{bounded flipwidth}_\infty \arrow[d] \\
\text{bounded expansion} \arrow[rr, "1\leq r<\infty"] \arrow[d, phantom, "="] & & \text{bounded flip-width} \\
\text{bounded copwidth}_r\ \forall r\in\mathbb{N} & &
\end{tikzcd}$$

Fact Bounded twin-width
$\downarrow$
Bounded flip-width.

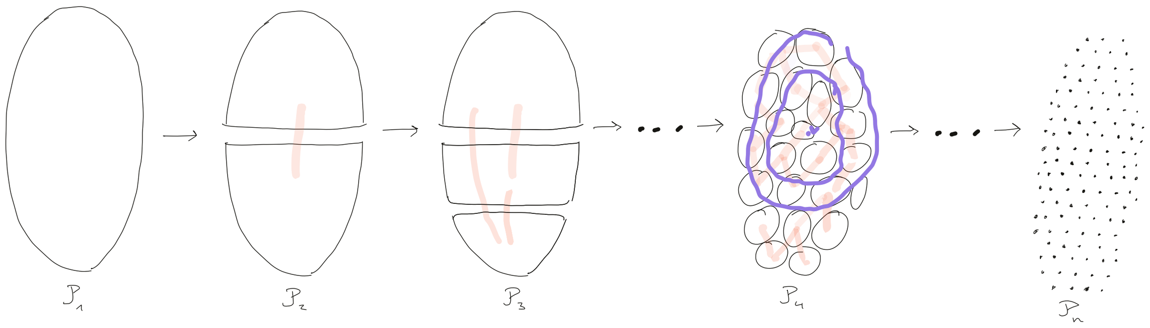
U is an (r,k)-hideout for runner if
for every k-flip G' of G:
Mark $v \in U$ red if v r-reaches $\leq k$ vertices of U in G'
Then # red vertices is $\leq k$.

Lemma If U is an (r,k)-hideout
then runner wins the radius-r Flipper game of width k:
$fw_r(G) > k$.

Lemma If U is well-linked and $|U| > 3k$
then U is an (∞,k)-hideout,
so $fw_\infty(G) > k$.

$fw_\infty(G) \leq k \Rightarrow$ no well-linked set of size $> 3k$ $\Rightarrow \text{rankwidth}(G) \leq 3k+1$.

Corrolary For a graph class $\mathcal{C}$,
$\text{rankwidth}(\mathcal{C}) < \infty \iff fw_\infty(\mathcal{C}) < \infty$



Invariant:

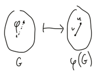
possible next moves of runner
$\cap$
ball of radius r in red graph

Properties

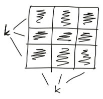
Theorem $\mathcal{C}$ – graph class. Then:
$\mathcal{C}$ has bounded expansion $\Leftrightarrow$ $\mathcal{C}$ is $K_{t,t}$-free and $\mathcal{C}$ has bounded flip-width

Theorem $\mathcal{C}$ – class of ordered graphs $G=(V,E,<)$.
$\mathcal{C}$ has bounded twin-width $\Leftrightarrow$ $\mathcal{C}$ has bounded flip-width.

Theorem If $\mathcal{C}$ has bounded flip-width
and $\varphi(x,y)$ – first order formula,
then $\varphi(\mathcal{C})$ has bounded flip-width



Corollary Classes of bounded flip-width are monadically NIP.



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\text{bounded expansion} \arrow[rr] \arrow[d] & & \text{bounded flip-width} \arrow[d] \\
\text{nowhere dense} \arrow[rr] & & ?\ \text{monadically NIP}
\end{tikzcd}$$

Fact (Nešetřil, Ossona de Mendez)
A hereditary graph class $\mathcal{C}$ is nowhere dense
$\Updownarrow$
$\text{wcol}_r(G) \leq O(|G|^\varepsilon)$ for all fixed $1 \leq r < \infty$ and $\varepsilon > 0$ and all $G \in \mathcal{C}$.

Def
A hereditary graph class $\mathcal{C}$ has almost bounded flip-width
$\Updownarrow$
$fw_r(G) \leq O(|G|^\varepsilon)$ for all fixed $1 \leq r < \infty$ and $\varepsilon > 0$ and all $G \in \mathcal{C}$.

Thank you
for your attention!

Properties

Theorem $\mathcal{C}$ – hereditory graph class. Then:
$\mathcal{C}$ is nowhere dense $\Leftrightarrow$
$\mathcal{C}$ is $K_{t,t}$-free and $\mathcal{C}$ has almost bounded flip-width

Theorem Structurally nowhere dense
$\Downarrow$
almost bounded twin-width

$\mathcal{C}$ – hereditary class of ordered graphs $G=(V,E,<)$

$\mathcal{C}$ has bounded twin-width
$\Leftrightarrow$ $\mathcal{C}$ has bounded flip-width.
$\Leftrightarrow$ $\mathcal{C}$ has almost bounded flip-width.

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\text{bounded expansion} \arrow[r] \arrow[d] & \text{bounded flip-width} \arrow[d] \\
\text{nowhere dense} \arrow[r] & \text{almost bounded flip-width} \\
 & \overset{?}{\|} \\
 & \text{monadically NIP}
\end{tikzcd}$$

Questions

- Is FO-model checking FPT on classes of (almost) bounded flip-width?

- Almost bounded flip-width
  $\overset{?}{\|}$
  monadic NIP (Shelah)

- Decompositions & obstructions for flip-width.

- existence of hideouts
  $\Updownarrow$
  unbounded flip-width ?

- combinatorial properties:
  - Strong Erdős-Hajnal,
  - χ- boundedness
  - linear neighborhood complexity