1 Just a warm-up

Let $P \in \mathbb{F}[X,Y]$. 

1. Case $\mathbb{F} = \mathbb{R}$. Show that if $P$ vanishes on $\mathbb{R}^2$, then $P = 0$.

2. Case $\mathbb{F}$ finite. Show that $P$ can vanish on $\mathbb{F}^2$ with $P \neq 0$.

3. Show that the Schwartz-Zippel lemma is sharp by providing some examples achieving the $d|S|^n - 1$ bound.

2 Alon’s combinatorial nullstellensatz

Let $P \in \mathbb{F}[X_1, \ldots, X_n], P \neq 0$, of degree $d$, and $S_1, \ldots, S_n$ be finite subsets of $\mathbb{F}$.

1. (Alon-Tarsi) Assume that $|S_i| > \deg_i(P)$ for all $i = 1, \ldots, n$, where $\deg_i(P)$ is the maximum degree of $x_i$ in $P$. Show that $P$ cannot vanish on $S_1 \times S_2 \times \ldots \times S_n$.

2. Assume $P$ vanishes on $S_1 \times \cdots \times S_n$ and let $g_i = \prod_{s \in S_i} (X_i - s)$. Show that there exist polynomials $h_1, \ldots, h_n \in \mathbb{F}[X_1, \ldots, X_n]$ such that $\deg(h_i) \leq d - |S_i|$ and

$$P = \sum_{i=1}^n h_ig_i.$$ 

3. Assume that $P$ contains a term $c \cdot X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ with $c \neq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = d$. Show that if $|S_i| > \alpha_i$ for all $i = 1, \ldots, n$, then $P$ does not vanish on $S_1 \times S_2 \times \cdots \times S_n$.

3 Alon-Füredi

Consider $H_n = \{0, 1\}^n$, the set of vertices of the $n$-cube of $\mathbb{R}^n$. Note that two hyperplanes can cover $H_n$, namely $x_1 = 0$ and $x_1 = 1$. A surprising result asserts that the minimum number of hyperplanes needed to cover $H_n \setminus \{0^n\}$, but not $0^n$, actually jumps to $n$.

1. Convince yourself of this fact in low dimension.

2. Assume for contradiction that there exist $m < n$ hyperplanes $a^{(i)} \cdot x = b^{(i)}$ for $i = 1, \ldots, m$ covering $H_n \setminus \{0^n\}$ but not $0^n$ (i.e. all $b^{(i)}$ are non-zero). Form a polynomial $P$ with degree $n$ which vanishes on $H_n$ and such that $X_1 \cdot X_2 \cdots X_n$ appears as a term with non-zero coefficient.

3. Conclude.