ENS de Lyon — Math Department | Master 1 — Spring 2018
Brownian Motion and Stochastic Processes | E. Jacob & M. Maazoun

Homework assignement: on the zero set of B v3

This homework assignment is not mandatory and won't be graded. However we recommend that you do it to prepare for the midterms.
You can have it corrected if you turn it in (in total or in part) whenever you get the chance. The earlier the better chance you'll have to receive your corrected copy before the midterms. A solution will be posted online at some point.
The exercises entitled "for your personal enjoyment" do not contain any probability. You can skip them if you want.

We denote by B a standard Brownian motion started at 0, and

$$Z = \{t \geq 0, B_t = 0\}.$$

In a previous exercise sheet, we showed that Z is almost surely a closed set with no isolated points. The point of the present problem is to understand how "big" Z is. A first approach is to compute its measure.

Exercise 1 — *Triviality.*
Show that almost surely, the Lebesgue measure of Z is 0.

This means that Z is not too big. But it's not too small either:

Exercise 2 — *For your personal enjoyment.*
Show that a nonempty complete metric space with no isolated points is uncountable.

Hence we need a finer framework to capture the exact "size" of Z. This will be *Hausdorff dimension.* Let (E, d) be a metric space. For $\alpha \geq 0$ and $A \subset E$, we define the *α-dimensional Hausdorff measure*[1] of A follows:

$$\mathcal{H}_\alpha(A) := \lim_{\delta \to 0} \left(\inf_{\substack{(U_i)_i \in \mathcal{P}(E)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \leq \delta \\ \bigcup_i U_i \supset A}} \left(\sum_{i \in \mathbb{N}} \mathrm{diam}(U_i)^\alpha \right) \right).$$

It is well defined because the lim is actually a sup, and verifies the following property:

Lemma 1. *Let $\alpha \in [0, \infty)$.If $\mathcal{H}_\alpha(A) < \infty$ then for $\beta > \alpha$ $\mathcal{H}_\beta(A) = 0$. If $\mathcal{H}_\alpha(A) > 0$ then for $\beta < \alpha$ $\mathcal{H}_\beta(A) = \infty$.*

[1]This is not a measure in our usual definition of the term. It is not necessarily σ-additive but always σ-subadditive ~~and finitely additive~~. On the other hand, it is defined for all sets and not just on the Borel σ-algebra. We call this an *outer measure.*

This tells us that there is a transition point $\alpha \in [0, \infty]$ where the Hausdorff measure jumps from ∞ to 0, and we want to call that point the Hausdorff dimension of A.

$$\dim_{\mathcal{H}}(A) := \sup\{\alpha, \mathcal{H}_\alpha(A) = \infty\} = \inf\{\alpha, \mathcal{H}_\alpha(A) = 0\}.$$

This α is the only dimension for which A admits a possibly non-trivial Hausdorff measure (but it may still be 0 or ∞ in some cases).
For instance, in $\mathbb{R}^d$, the d-dimensional Hausdorff measure is equal to the Lebesgue measure (you probably constructed the Lebesgue measure this way), and open sets have necessarily Hausdorff dimension d. Of course sets with 0 Lebesgue measure (like Z), might have a strictly smaller Hausdorff dimension.

Exercise 3 — *For your personal enjoyment.*
Prove all unproved claims above. ~~Show also that the Haussdorf measure has good properties w.r.t. finite partitions and scaling, deduce~~ Guess the Hausdorff dimension of your favorite self-similar fractal, and check your answer on this (very cool) Wikipedia page: `https://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension`.

Our goal now is to compute the Haussdorf dimension of Z. We start with the upper bound, which requires us to exhibit coverings by small sets with arbitrarily small α-size. To that end we first need to compute the probability that a zero exists in some small interval, through the following exercise which is of independent interest.

Exercise 4 — *Last* 0 *before time* 1 *(Second arcsine Law).*
Compute the cumulative distribution function of G_1, where $G_u := \sup(Z \cap [0, u])$. Deduce the value of $\mathbb{P}(Z \cap [x, x + \epsilon] \neq \emptyset)$ for $x \geq 0, \epsilon > 0$, and deduce the upper estimate $2\sqrt{\epsilon/(x + \epsilon)}$ for this probability.

From there we can get our upper bound.

Exercise 5 — *Upper bound.*
For $n \geq 1$ construct the random covering C_n of $Z \cap [0, 1]$ by taking the intervals of the form $[k2^{-n}, (k + 1)2^{-n}]$ for $0 \leq k \leq 2^n - 1$ that intersect Z.

(1) Show that $\mathbb{E}[\sum_{I \in C_n} \mathrm{diam}(I)^\alpha]$ goes to 0 as $n \to \infty$ when $\alpha > 1/2$.
(2) Deduce that almost surely you can find arbitrarily α-small coverings of $Z \cap [0, 1]$ arbitrarily far away in the sequence C_n
(3) Deduce that almost surely $\dim_{\mathcal{H}}(Z \cap [0, 1]) \leq 1/2$

The lower bound is always more tricky, as we need to show that for $\alpha < 1/2$, *all* coverings by small sets must be bounded away from 0. A way to do this is to compare the Hausdorff measure with some existing measure, as stated in the following theorem:

Theorem 1 (Mass distribution principle)**.** *Suppose that (E, d) is a metric space endowed with some nonzero Borelian measure μ, and $C > 0$, $\delta > 0$, $\alpha \geq 0$ such that for every closed*

$X \subset E$ with diameter $\leq \delta$, we have $\mu(X) \leq C\,\mathrm{diam}(X)^\alpha$. Then $\mathcal{H}_\alpha(E) \geq \mu(E)/C$ and hence $\dim_{\mathcal{H}}(E) \geq \alpha$.

Now we can prove our lower bound.

Exercise 6 — *Lower bound.*

(1) Prove Theorem 1.
(2) Show that Z is distributed like the record set $R = \{t \geq 0, B_t = B_t^*\}$ of B.
(3) Find a natural measure that is supported by R. *Hint :* find a weakly increasing function that does not increase outside of R...
(4) Use the mass distribution principle to show that almost surely $\dim_{\mathcal{H}}(Z) \geq 1/2 - \alpha$ for every $\alpha > 0$.
(5) Conclude on the Hausdorff dimension of Z.